

$D = 4$ pure (anti-)de Sitter supergravity revisited

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Abstract: By considering that $N = 2$ during the supersymmetrization process and by imposing the $N = 1$ condition only at the end, $D = 4, N = 1$ pure anti-de Sitter supergravity is revisited in an heterodox way that can be used to derive $D = 4$ pure de Sitter supergravity without having to introduce any other field than those of the graviton and the gravitino.

“One should not desist from pursuing to the end the path of the relativistic field theory.”
A. Einstein

“A great deal of my work is just playing with equations and seeing what they give.”
P. Dirac

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1 Introduction

Supergravity is a wonderful achievement at the crossroads between the works of A. Einstein and P. Dirac [1–4]. Noteworthily, almost since its discovery forty-five years ago, $D = 4, N = 1$ supergravity is known to be anti-de Sitter with a negative cosmological constant when only the graviton bosonic field e_μ^m and the gravitino fermionic field ψ_μ are considered [5]. It is only recently that $D = 4, N = 1$ pure de Sitter supergravity with a positive cosmological constant has been derived by adding a nilpotent Goldstino fermionic field χ to obtain the local supersymmetry and then by eliminating it through supersymmetry breaking [6].

Given the no-go theorems on the subject [7, 8], it is clear that hoping to derive $D = 4$ pure de Sitter supergravity in a different way than [6] requires an heterodox approach. This paper presents such an approach that consists in doubling the usual number of Majorana spinors for both the gravitino field and the supersymmetry parameter, which allows to construct spinor-ansatzes that cancel the usual quartic fermion terms appearing in the supersymmetrization process.

For motivation to derive pure de Sitter supergravity, the reader is referred to the Introduction of [6].

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2 A reminder on spinors, Lorentz transformations, local Lorentz derivative, and spinor bilinears

2.1 Spinors

This section follows Sec. 3.2 of [4]. Given any spinor $\chi \equiv (\chi)_\alpha$ whose components are four anti-commuting Grassmann variables, its Dirac conjugate $\bar{\chi}^{(\text{Dirac})} \equiv (\chi)^\alpha$ and its charge conjugate $\chi^C \equiv (\chi^C)_\alpha$ are defined by¹

$$\bar{\chi}^{(\text{Dirac})} \equiv i\chi^\dagger \gamma^0, \quad (1)$$

$$\chi^C \equiv i\gamma^0 C^\dagger \chi^*, \quad (2)$$

where C is the charge conjugation matrix having the following properties²

$$\begin{aligned} C^\dagger &= C^{-1}, \quad C^T = -C \implies CC^* = C^*C = -\mathbb{1}, \\ (C\gamma_*)^T &= -C\gamma_*. \\ (C\gamma^m)^T &= C\gamma^m, \\ (C\gamma^{mn})^T &= C\gamma^{mn}, \\ (C\gamma^{mnr})^T &= -C\gamma^{mnr}, \\ (C\gamma^{mnrs})^T &= -C\gamma^{mnrs}. \end{aligned} \quad (3)$$

From the definitions (1),(2) and the properties (3), one can verify that

$$\overline{\chi^C}^{(\text{Dirac})} \equiv i(\chi^C)^\dagger \gamma^0 = \chi^T C, \quad (4)$$

which can be used to define the Majorana conjugate of any spinor χ by

$$\bar{\chi}^{(\text{Majorana})} \equiv \chi^T C. \quad (5)$$

In this paper, the Majorana conjugate (5) will be used and not the Dirac conjugate (1) for it simplifies the calculations by not involving complex conjugation.

2.2 Majorana spinors

Supergravity uses Majorana spinors that are defined by the condition

$$\chi^C = \chi. \quad (6)$$

From (4),(5) one can see that the Dirac conjugate of a Majorana spinor is equal to its Majorana conjugate

$$\chi^C = \chi \implies \bar{\chi}^{(\text{Dirac})} \equiv \bar{\chi}^{(\text{Majorana})}. \quad (7)$$

¹The conventions of this paper are those of [4]: the metric signature is $(-+++)$; the four γ -matrices are defined by $\gamma^m \gamma^n + \gamma^n \gamma^m = 2\eta^{mn}\mathbb{1}$ where $\mathbb{1}$ is the unit matrix $\Rightarrow (\gamma^0)^2 = -\mathbb{1}$ and $(\gamma^k)^2 = \mathbb{1}$ with $k = 1, 2, 3$; $(\gamma^m)^\dagger = \gamma^0 \gamma^m \gamma^0 \Rightarrow (\gamma^0)^\dagger = -\gamma^0$, $(\gamma^k)^\dagger = \gamma^k$; $\gamma_m \equiv \eta_{mn} \gamma^n \Rightarrow \gamma_0 = -\gamma^0$ and $\gamma_k = \gamma^k$; $\gamma_* \equiv -i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = i\gamma_0 \gamma_1 \gamma_2 \gamma_3 \Rightarrow (\gamma_*)^2 = \mathbb{1}$, $(\gamma_*)^\dagger = \gamma_*$ and $\gamma_* \gamma^m = -\gamma^m \gamma_*$; $\gamma^{mn} \equiv \gamma^{[m} \gamma^{n]} = \frac{1}{2}(\gamma^m \gamma^n - \gamma^n \gamma^m)$ antisymmetric with strength one, and so on for γ^{mnr} and γ^{mnrs} ; $\epsilon^{\mu\nu\rho\sigma} \equiv e \epsilon^{mnrs} e_m^\mu e_n^\nu e_r^\rho e_s^\sigma \Rightarrow \epsilon_{\mu\nu\rho\sigma} = e^{-1} \epsilon_{mnrs} e_\mu^m e_\nu^n e_\rho^r e_\sigma^s$ with $\epsilon_{0123} = +1 = -\epsilon^{0123}$.

²The four most used representations of the γ -matrices are given in Appendix A.

2.3 Lorentz transformations

The Lorentz transformations of a vector V^m and a spinor χ are defined by³

$$\delta_L V^m = -\lambda^m{}_k V^k \iff \delta_L V_m = -\lambda_m{}^k V_k, \quad (8)$$

$$\delta_L \chi = -\frac{1}{4} \lambda_{ab} \gamma^{ab} \chi \iff \delta_L \bar{\chi} = \frac{1}{4} \lambda_{ab} \bar{\chi} \gamma^{ab}, \quad (9)$$

where λ_{ab} is an infinitesimal Lorentz transformation that is antisymmetric $\lambda_{ab} = -\lambda_{ba}$

2.4 Weyl projections

Any spinor χ can be split into its left and right Weyl projections that are also spinors⁴

$$P_L \chi \equiv \frac{1}{2} (\mathbb{1} + \gamma_*) \chi, \quad (10)$$

$$P_R \chi \equiv \frac{1}{2} (\mathbb{1} - \gamma_*) \chi, \quad (11)$$

with obviously $\chi = P_L \chi + P_R \chi$.

In $D = 4$ it is known that a spinor χ cannot be a Majorana and a Weyl spinor at the same time⁵: $\chi = \chi^C \Rightarrow \chi \neq P_L \chi$ and $\chi \neq P_R \chi$.

2.5 Local Lorentz derivative

Supergravity uses the local Lorentz derivative which acts on Lorentz local frame (Latin) indices and spinor indices but not on general coordinate (Greek) indices. The local Lorentz derivative of a vector V^m and a spinor χ are defined by⁶

$$D_\mu V^m = \partial_\mu V^m + \omega_\mu{}^m{}_k V^k \implies D_\mu V_m = \partial_\mu V_m + \omega_{\mu m}{}^k V_k, \quad (12)$$

$$D_\mu \chi = \partial_\mu \chi + \frac{1}{4} \omega_{\mu ab} \gamma^{ab} \chi \implies D_\mu \bar{\chi} = \partial_\mu \bar{\chi} - \frac{1}{4} \omega_{\mu ab} \bar{\chi} \gamma^{ab}, \quad (13)$$

where $\omega_{\mu mn}$ is the spin connection that is antisymmetric $\omega_{\mu mn} = -\omega_{mn\mu}$.

2.6 Spinor bilinears

In the two previous sections, it is implied that the γ -matrices and C are numerical matrices on which the Lorentz transformation and the Lorentz derivative do not act⁷:

$$\delta_L \gamma^m = 0, \quad D_\mu \gamma^m = 0, \quad (14)$$

$$\delta_L \gamma_* = 0, \quad D_\mu \gamma_* = 0, \quad (15)$$

$$\delta_L C = 0, \quad D_\mu C = 0. \quad (16)$$

³The expression for $\delta_L \bar{\chi}$ follows directly from (5) and the properties (3): it is easy to verify that $\overline{(\delta_L \chi)} \equiv (\delta_L \chi)^T C = \delta_L (\chi^T C) = \delta_L \bar{\chi}$.

⁴Needless to say that a Weyl projection satisfies (9) thanks to the property $\gamma_* \gamma^{ab} = \gamma^{ab} \gamma_*$.

⁵This is easy to verify from the explicit expression for a Majorana spinor in the Weyl representation of the γ -matrices given in Appendix A.

⁶The expression for $D_\mu \bar{\chi}$ follows directly from (5) and the properties (3): it is easy to verify that $\overline{(D_\mu \chi)} \equiv (D_\mu \chi)^T C = D_\mu (\chi^T C) = D_\mu \bar{\chi}$.

⁷See Sec. 8.3 of [4]. Note that this is not always clearly stated in the literature.

The Lorentz transformation (9) together with the left part of (14),(16) allow to create spinor bilinears that transform as tensorial entities. For instance, $\bar{\chi}\xi$ transform as a scalar: $\delta_L(\bar{\chi}\xi) = \delta_L\bar{\chi}\xi + \bar{\chi}\delta_L\xi = \frac{1}{4}\lambda_{ab}\bar{\chi}\gamma^{ab}\xi - \frac{1}{4}\lambda_{ab}\bar{\chi}\gamma^{ab}\xi = 0$.

Using the relation $\gamma^{ab}\gamma^m - \gamma^m\gamma^{ab} = 2\eta^{bm}\gamma^a - 2\eta^{am}\gamma^b$ one can see that $\bar{\chi}\gamma^m\xi$ transform as a vector: $\delta_L(\bar{\chi}\gamma^m\xi) = \delta_L\bar{\chi}\gamma^m\xi + \bar{\chi}\gamma^m\delta_L\xi = \frac{1}{4}\lambda_{ab}\bar{\chi}\gamma^{ab}\gamma^m\xi - \frac{1}{4}\lambda_{ab}\bar{\chi}\gamma^m\gamma^{ab}\xi = \frac{1}{4}\lambda_{ab}\bar{\chi}(\gamma^{ab}\gamma^m - \gamma^m\gamma^{ab})\xi = \frac{1}{2}\lambda_{ab}\bar{\chi}(\eta^{bm}\gamma^a - \eta^{am}\gamma^b)\xi = \frac{1}{2}\lambda_a{}^m\bar{\chi}\gamma^a\xi - \frac{1}{2}\lambda^m{}_b\bar{\chi}\gamma^b\xi = -\lambda^m{}_k(\bar{\chi}\gamma^k\xi)$.

Similarly, one can verify that the spinor bilinears $\bar{\chi}\gamma^{mn}\chi$, $\bar{\chi}\gamma^{mnr}\chi$ and $\bar{\chi}\gamma^{mnrs}\chi$ transform as tensors.

3 Spinor-ansatzes

Starting from two Majorana spinors $\psi^1 \equiv (\psi^1)^C$ and $\psi^2 \equiv (\psi^2)^C$, let's construct the spinor-ansatz Ψ^i and its Majorana conjugate $\bar{\Psi}^i$ based on their Weyl projections⁸

$$\begin{aligned}\Psi^1 &\equiv \frac{1}{\sqrt{2}}P_L\psi^1, \quad \bar{\Psi}^1 \equiv (\Psi^1)^TC = \frac{1}{\sqrt{2}}P_L\bar{\psi}^1, \\ \Psi^2 &\equiv \frac{1}{\sqrt{2}}P_R\psi^2, \quad \bar{\Psi}^2 \equiv (\Psi^2)^TC = \frac{1}{\sqrt{2}}P_R\bar{\psi}^2, \\ \Psi^3 &\equiv (\Psi^1)^C = \frac{1}{\sqrt{2}}P_R\psi^1, \quad \bar{\Psi}^3 \equiv (\Psi^3)^TC = \frac{1}{\sqrt{2}}P_R\bar{\psi}^1, \\ \Psi^4 &\equiv (\Psi^2)^C = \frac{1}{\sqrt{2}}P_L\psi^2, \quad \bar{\Psi}^4 \equiv (\Psi^4)^TC = \frac{1}{\sqrt{2}}P_L\bar{\psi}^2.\end{aligned}\tag{17}$$

From two spinor-ansatzes ε^i , Ψ^i let's also define the following spinor-ansatz bilinear which is real by construction⁹

$$\mathbf{M}^{ij}(\bar{\varepsilon}^i N \Psi^j), \tag{18}$$

where N is any matrix obtained from products of γ^m and $i\gamma_*$ and where \mathbf{M} is a 4x4 block diagonal numerical matrix

$$\mathbf{M} = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix},$$

in which M is a 2x2 numerical matrix whose components are real numbers.

4 The derivation framework

The derivation framework used to revisit $D = 4$ pure (anti-)de Sitter supergravity makes use of three 4x4 block diagonal numerical matrices \mathbf{S}_1 , \mathbf{S}_2 , \mathbf{M}_1 whose components are real numbers and four conventional real constants k_1, k_2, k_3, k_4 to be specified in the next section. In this section it is only supposed that the two 4x4 matrices \mathbf{S} are symmetric: $\mathbf{S} = \mathbf{S}^T$.

The action considered is similar to the one of $D = 4, N = 2$ pure anti-de Sitter supergravity¹⁰ for the terms based only on the graviton real bosonic field e_μ^m , the cosmological real constant Λ , and the gravitino complex fermionic field $\Psi_\mu^i \equiv (\Psi_\mu^i)_\alpha$.

⁸The consistency of this construction is proven in Appendix B.1.

⁹This is proven in Appendix C.

¹⁰See Sec. 2.8 of [3].

The action is the sum of the Einstein-Hilbert term, the cosmological constant term, the Rarita-Schwinger term, and one gauged term:

$$S = S_{\text{EH}} + S_{\Lambda} + S_{\text{RS}} + S_{g\Lambda}, \quad (19)$$

where¹¹

$$S_{\text{EH}} = k_1 \int dx^4 e e_m^\mu e_n^\nu R_{\mu\nu}^{mn}, \quad (20)$$

$$S_{\Lambda} = \pm 2k_1 \int dx^4 e \Lambda, \quad (21)$$

$$S_{\text{RS}} = -k_2 \int dx^4 e S_1^{ij} (\bar{\Psi}_\mu^i \gamma^{\mu\nu\rho} D_\nu \Psi_\rho^j), \quad (22)$$

$$S_{g\Lambda} = +2k_2 k_4 \sqrt{\Lambda} \int dx^4 e S_2^{ij} (\bar{\Psi}_\mu^i \gamma^{\mu\nu} \Psi_\nu^j), \quad (23)$$

with

$$R_{\mu\nu}^{mn} \equiv \partial_\mu \omega_\nu^{mn} - \partial_\nu \omega_\mu^{mn} + \omega_\mu^m{}_r \omega_\nu^{rn} - \omega_\nu^m{}_r \omega_\mu^{rn}, \quad (24)$$

$$D_\mu \Psi_\nu^i \equiv \partial_\mu \Psi_\nu^i + \frac{1}{4} \omega_{\mu mn} \gamma^{mn} \Psi_\nu^i \implies D_\mu \bar{\Psi}_\nu^i = \partial_\mu \bar{\Psi}_\nu^i - \frac{1}{4} \omega_{\mu mn} \bar{\Psi}_\nu^i \gamma^{mn}. \quad (25)$$

With the conventions given in footnote 1, the plus (resp. minus) sign of the cosmological constant term (21) corresponds to pure anti-de Sitter (resp. de Sitter) supergravity.

Using the so-called 1.5 order formalism, the goal is to show that the action (19)-(23) is invariant $\delta S = 0$ under the following local supersymmetry transformations¹² involving the spinor-ansatz $\varepsilon^i \equiv (\varepsilon^i)_\alpha$ as supersymmetry parameter¹³:

$$\delta e_\mu^m = k_3 S_1^{ij} (\bar{\varepsilon}^i \gamma^m \Psi_\mu^j) \implies \delta e_\mu^\mu = -k_3 S_1^{ij} (\bar{\varepsilon}^i \gamma^\mu \Psi_m^j) \implies \delta e = k_3 e S_1^{ij} (\bar{\varepsilon}^i \gamma^\rho \Psi_\rho^j), \quad (26)$$

$$\delta \Psi_\mu^i = D_\mu \varepsilon^i + k_4 \sqrt{\Lambda} M_1^{ih} \gamma_\mu \varepsilon^h \implies \delta \bar{\Psi}_\mu^i = D_\mu \bar{\varepsilon}^i - k_4 \sqrt{\Lambda} M_1^{ih} \bar{\varepsilon}^h \gamma_\mu. \quad (27)$$

As stated in [2] the 1.5 order formalism is nothing else than the Palatini trick of general relativity extended to supergravity. The spin connection ω_μ^{mn} is treated as an independent field and one imposes that the variation of the action (19)-(23) with respect to it vanishes:

$$\delta_\omega S = \delta_\omega S_{\text{EH}} + \delta_\omega S_{\text{RS}} = 0, \quad (28)$$

where¹⁴

$$\delta_\omega S_{\text{EH}} = k_1 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnrs} \delta \omega_\mu^{mn} e_\nu^r D_\rho e_\sigma^s, \quad (29)$$

$$\delta_\omega S_{\text{RS}} = -\frac{k_2}{4} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnrs} \delta \omega_\mu^{mn} e_\nu^r S_1^{ij} (\bar{\Psi}_\rho^i \gamma^s \Psi_\sigma^j), \quad (30)$$

¹¹As usual in supergravity: $e = \det e_\mu^m$; $\epsilon_{\mu\nu\rho\sigma} \equiv e^{-1} \epsilon_{mnrs} e_\mu^m e_\nu^r e_\rho^s e_\sigma^t$, $\epsilon^{\mu\nu\rho\sigma} \equiv e \epsilon^{mnrs} e_\mu^m e_\nu^r e_\rho^s e_\sigma^t$ with $\epsilon_{0123} = +1 = -\epsilon_{0123}^{0123}$ and $\epsilon^{\mu\nu\rho\sigma} \epsilon_{mnrs} e_\rho^s e_\sigma^t = -2e (e_m^\mu e_n^\nu - e_n^\mu e_m^\nu)$; $\gamma^{\mu\nu\rho} \equiv \gamma^{mnrs} e_m^\mu e_n^\nu e_r^\rho = -e^{-1} \epsilon^{\mu\nu\rho\sigma} i \gamma_* \gamma_\sigma$ with $\gamma_\sigma \equiv e_\sigma^m \gamma_m$.

¹²The expression for $\delta \bar{\Psi}_\mu^i$ follows directly from (6) and the properties (3): it is easy to verify that $(\delta \bar{\Psi}_\mu^i) = (\delta \Psi_\mu^i)^T C = \delta (\Psi_\mu^i)^T C = \delta \bar{\Psi}_\mu^i$.

¹³The consistency of the local supersymmetry transformation (27) is proven in Appendix B.2.

¹⁴See Appendix D for the detailed calculations.

which leads to the equation

$$D_{[\rho} e_{\sigma]}^s = \frac{k_2}{4k_1} \mathbf{S}_1^{ij} \lambda (\bar{\Psi}_{\rho}^i \gamma^s \Psi_{\sigma}^j). \quad (31)$$

This equation for ω_{μ}^{mn} can be solved to find

$$\omega_{\mu}^{mn} = \omega_{\mu}^{mn}(e) + K_{\mu}^{mn}(\Psi), \quad (32)$$

where $\omega_{\mu}^{mn}(e)$ is the torsionless spin connection

$$\omega_{\mu}^{mn}(e) = \frac{1}{2} e^{m\rho} (\partial_{\mu} e_{\rho}^n - \partial_{\rho} e_{\mu}^n) - \frac{1}{2} e^{n\rho} (\partial_{\mu} e_{\rho}^m - \partial_{\rho} e_{\mu}^m) - \frac{1}{2} e^{m\rho} e^{n\sigma} (\partial_{\rho} e_{\sigma}^r - \partial_{\sigma} e_{\rho}^r) e_{r\mu}, \quad (33)$$

and $K_{\mu}^{mn}(\Psi)$ is the so-called contortion tensor given by

$$K_{\mu}^{mn}(\Psi) = \frac{k_2}{4k_1} \mathbf{S}_1^{ij} \lambda (\bar{\Psi}_{\mu}^i \gamma^m \Psi_{\nu}^j - \bar{\Psi}_{\mu}^i \gamma^n \Psi_{\nu}^j + \bar{\Psi}_{\mu}^i \gamma^m \Psi_{\nu}^j). \quad (34)$$

Following the spirit of the 1.5 order formalism, the result (31) shall be taken into account in the next variations. It is proven in Appendix D that the next variations give

$$\begin{aligned} \delta_e S_{\text{EH}} &= -\frac{k_1 k_3}{2} \int dx^4 \overbrace{\epsilon^{\mu\nu\rho\sigma} \epsilon_{mnrs} R_{\mu\nu}^{mn} e_{\rho}^r \mathbf{S}_1^{ij} (\bar{\varepsilon}^i \gamma^s \Psi_{\sigma}^j)}^a. \\ \delta_e S_{\Lambda} &= \pm 2k_1 k_3 \Lambda \int dx^4 e \mathbf{S}_1^{ij} (\bar{\varepsilon}^i \gamma^{\mu} \Psi_{\mu}^j). \\ \delta_e S_{\text{RS}} &= -ik_2 k_3 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\varepsilon}^i \gamma_m \Psi_{\mu}^j) \mathbf{S}_1^{kl} (\bar{\Psi}_{\nu}^k \gamma_* \gamma^m D_{\rho} \Psi_{\sigma}^l). \\ \delta_e S_{g\Lambda} &= -2ik_2 k_3 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\varepsilon}^i \gamma_m \Psi_{\mu}^j) \mathbf{S}_2^{kl} (\bar{\Psi}_{\nu}^k \gamma_* \gamma^m \gamma_{\rho} \Psi_{\sigma}^l). \\ \delta_{\Psi} S_{\text{RS}} &= +\frac{k_2}{4} \int dx^4 \overbrace{\epsilon^{\mu\nu\rho\sigma} \epsilon_{mnrs} R_{\mu\nu}^{mn} e_{\rho}^r \mathbf{S}_1^{ij} (\bar{\varepsilon}^i \gamma^s \Psi_{\sigma}^j)}^a + 4k_2 k_4 \sqrt{\Lambda} \int dx^4 e \mathbf{S}_1^{jh} \mathbf{M}_1^{hi} (\bar{\varepsilon}^i \gamma^{\mu\nu} D_{\mu} \Psi_{\nu}^j) \\ &\quad - \frac{i(k_2)^2}{4k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\varepsilon}^i \gamma_* \gamma_m D_{\mu} \Psi_{\nu}^j) \mathbf{S}_1^{kl} (\bar{\Psi}_{\rho}^k \gamma^m \Psi_{\sigma}^l) \\ &\quad - \frac{i(k_2)^2 k_4 \sqrt{\Lambda}}{4k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{jh} \mathbf{M}_1^{hi} (\bar{\varepsilon}^i \gamma_* \gamma_{\mu} \gamma_m \Psi_{\nu}^j) \mathbf{S}_1^{kl} (\bar{\Psi}_{\rho}^k \gamma^m \Psi_{\sigma}^l). \\ \delta_{\Psi} S_{g\Lambda} &= -4k_2 k_4 \sqrt{\Lambda} \int dx^4 e \mathbf{S}_2^{ij} (\bar{\varepsilon}^i \gamma^{\mu\nu} D_{\mu} \Psi_{\nu}^j) - 12k_2 (k_4)^2 \Lambda \int dx^4 e \mathbf{S}_2^{jh} \mathbf{M}_1^{hi} (\bar{\varepsilon}^i \gamma^{\mu} \Psi_{\mu}^j) \\ &\quad - \frac{i(k_2)^2 k_4 \sqrt{\Lambda}}{k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\bar{\varepsilon}^i \gamma_* \Psi_{\mu}^j) \mathbf{S}_1^{kl} (\bar{\Psi}_{\nu}^k \gamma_{\rho} \Psi_{\sigma}^l) \\ &\quad + \frac{i(k_2)^2 k_4 \sqrt{\Lambda}}{k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\bar{\varepsilon}^i \gamma_* \gamma_{\mu} \gamma_m \Psi_{\nu}^j) \mathbf{S}_1^{kl} (\bar{\Psi}_{\rho}^k \gamma^m \Psi_{\sigma}^l). \end{aligned} \quad (35)$$

In order to partly fulfill the supersymmetric condition $\delta S = 0$, the terms (a),(b),(c) of (35) give¹⁵

$$k_3 = \frac{k_2}{2k_1}, \quad k_4 = \frac{1}{2\sqrt{3}}, \quad (36)$$

¹⁵Note that k_2 is independent of k_1 . It is usual to set $k_1 = k_2 = \frac{1}{2\kappa^2} \Rightarrow k_3 = \frac{1}{2}$.

and¹⁶

$$(\mathbf{M}_1)^2 = \pm \mathbf{I} , \quad \mathbf{S}_2 = \mathbf{S}_1 \mathbf{M}_1 . \quad (37)$$

In order to completely fulfill the supersymmetric condition $\delta S = 0$, it is also proven in Appendix D that the remaining terms of (35) lead to

$$\begin{aligned} 0 &= -\frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{jk} \mathbf{S}_1^{il}) (\bar{\varepsilon}^i \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_* D_\rho \Psi_\sigma^l) \\ &\quad + \frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{jk} \mathbf{S}_1^{il}) (\bar{\varepsilon}^i \gamma_* \Psi_\mu^j) (\bar{\Psi}_\nu^k D_\rho \Psi_\sigma^l) \\ &\quad + \frac{i(k_2)^2}{4k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{jk} \mathbf{S}_1^{il}) (\bar{\varepsilon}^i D_\mu \Psi_\nu^j) (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^l) \\ &\quad - \frac{i(k_2)^2}{4k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{jk} \mathbf{S}_1^{il}) (\bar{\varepsilon}^i \gamma_* D_\mu \Psi_\nu^j) (\bar{\Psi}_\rho^k \Psi_\sigma^l) , \end{aligned} \quad (38)$$

and

$$\begin{aligned} 0 &= -\frac{i(k_2)^2 \sqrt{\Lambda}}{4k_1 \sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_2^{jl} - \mathbf{S}_1^{jk} \mathbf{S}_2^{il} - \mathbf{S}_1^{il} \mathbf{S}_2^{jk} + \mathbf{S}_1^{jl} \mathbf{S}_2^{ik}) (\bar{\varepsilon}^i \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \Psi_\sigma^l) \\ &\quad + \frac{i(k_2)^2 \sqrt{\Lambda}}{4k_1 \sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_2^{jl} - \mathbf{S}_1^{jk} \mathbf{S}_2^{il} + \mathbf{S}_1^{il} \mathbf{S}_2^{jk} - \mathbf{S}_1^{jl} \mathbf{S}_2^{ik}) (\bar{\varepsilon}^i \gamma_* \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_\rho \Psi_\sigma^l) \\ &\quad + \frac{i(k_2)^2 \sqrt{\Lambda}}{8k_1 \sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (2\mathbf{S}_1^{ik} \mathbf{S}_2^{jl} - 2\mathbf{S}_1^{il} \mathbf{S}_2^{jk} - \mathbf{S}_1^{jl} \mathbf{S}_2^{ik} + \mathbf{S}_1^{jk} \mathbf{S}_2^{il}) (\bar{\varepsilon}^i \gamma_\mu \Psi_\nu^j) (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^l) \\ &\quad - \frac{i(k_2)^2 \sqrt{\Lambda}}{8k_1 \sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (2\mathbf{S}_1^{ik} \mathbf{S}_2^{jl} - 2\mathbf{S}_1^{il} \mathbf{S}_2^{jk} + \mathbf{S}_1^{jl} \mathbf{S}_2^{ik} - \mathbf{S}_1^{jk} \mathbf{S}_2^{il}) (\bar{\varepsilon}^i \gamma_* \gamma_\mu \Psi_\nu^j) (\bar{\Psi}_\rho^k \Psi_\sigma^l) . \end{aligned} \quad (39)$$

Each line of the conditions (38)-(39) contains one spinor-ansatz bilinear based on $\bar{\varepsilon}^i \Psi_\mu^j$, $\bar{\varepsilon}^i \gamma_* \Psi_\mu^j$, $\bar{\Psi}_\rho^k \Psi_\sigma^l$, $\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^l$ that is anti-symmetric in i,j or k,l . It is easy to see that these spinor-ansatz bilinears vanish¹⁷ by construction of (17) and therefore that the conditions (38)-(39) are satisfied.

At this point, we have thus proven that the action (19)-(23) is locally supersymmetric under the local supersymmetry transformations (26)-(27) up to the conditions (37).

5 $D = 4$ pure anti-de Sitter supergravity

In this section the plus sign of the cosmological constant term (21) is considered. The following 4x4 matrices satisfy the conditions (37) when it is a plus sign in the first condition:

$$\begin{aligned} \mathbf{S}_1 &= \begin{pmatrix} S_1 & 0 \\ 0 & S_1 \end{pmatrix} \text{ with } S_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \\ \mathbf{M}_1 &= \begin{pmatrix} M_1 & 0 \\ 0 & M_1 \end{pmatrix} \text{ with } M_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \\ \mathbf{S}_2 &= \mathbf{S}_1 \mathbf{M}_1 = \begin{pmatrix} S_2 & 0 \\ 0 & S_2 \end{pmatrix} \text{ with } S_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \end{aligned} \quad (40)$$

¹⁶Note that \mathbf{I} is the unit matrix.

¹⁷Needless to say that this is due to the fact that $(1 + \gamma_*)(1 - \gamma_*) = 0$ for $(\gamma_*)^2 = 1$.

When imposing the $N = 1$ condition

$$\psi_\mu^1 = \psi_\mu^2 \equiv \psi_\mu, \quad (41)$$

it is easy to see from the construction (17) that $\Psi_\mu^1 + \Psi_\mu^2 = \Psi_\mu^3 + \Psi_\mu^4 = \frac{1}{\sqrt{2}}\psi_\mu$. One obtains therefore without surprise the action for $D = 4, N = 1$ pure anti-de Sitter supergravity¹⁸:

$$S = k_1 \int dx^4 e e_m^\mu e_n^\nu R_{\mu\nu}^{mn} + 2k_1 \int dx^4 e \Lambda - k_2 \int dx^4 e (\bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho) + \frac{k_2 \sqrt{\Lambda}}{\sqrt{3}} \int dx^4 e (\bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu), \quad (42)$$

with

$$\begin{aligned} R_{\mu\nu}^{mn} &\equiv \partial_\mu \omega_\nu^{mn} - \partial_\nu \omega_\mu^{mn} + \omega_\mu^m{}_r \omega_\nu^{rn} - \omega_\nu^m{}_r \omega_\mu^{rn}, \\ D_\mu \psi_\nu &\equiv \partial_\mu \psi_\nu + \frac{1}{4} \omega_{\mu mn} \gamma^{mn} \psi_\nu \implies D_\mu \bar{\psi}_\nu = \partial_\mu \bar{\psi}_\nu - \frac{1}{4} \omega_{\mu mn} \bar{\psi}_\nu \gamma^{mn}, \\ D_{[\rho} e_{\sigma]}^s &= \frac{k_2}{4k_1} \lambda(\bar{\psi}_\rho \gamma^s \psi_\sigma), \\ \omega_\mu^{mn} &= \frac{1}{2} e^{m\rho} (\partial_\mu e_\rho^n - \partial_\rho e_\mu^n) - \frac{1}{2} e^{n\rho} (\partial_\mu e_\rho^m - \partial_\rho e_\mu^m) - \frac{1}{2} e^{m\rho} e^{n\sigma} (\partial_\rho e_\sigma^r - \partial_\sigma e_\rho^r) e_{r\mu} + K_\mu^{mn}(\psi), \\ K_\mu^{mn}(\psi) &= \frac{k_2}{4k_1} \lambda(\bar{\psi}_\mu \gamma^m \psi^n - \bar{\psi}_\mu \gamma^n \psi^m + \bar{\psi}^m \gamma_\mu \psi^n). \end{aligned}$$

The $N = 1$ condition (41) can be imposed at the begining of the supersymmetrization process, which means that the action (42) is invariant $\delta S = 0$ under the following local supersymmetry transformations obtained from (26)-(27) when applying (41) with the matrices (40):

$$\delta e_\mu^m = \frac{k_2}{2k_1} (\bar{\epsilon} \gamma^m \psi_\mu) \implies \delta e_m^\mu = -\frac{k_2}{2k_1} (\bar{\epsilon} \gamma^\mu \psi_m) \implies \delta e = \frac{k_2}{2k_1} e (\bar{\epsilon} \gamma^\rho \psi_\rho), \quad (43)$$

$$\delta \psi_\mu = D_\mu \epsilon + \frac{\sqrt{\Lambda}}{2\sqrt{3}} \gamma_\mu \epsilon \implies \delta \bar{\psi}_\mu = D_\mu \bar{\epsilon} - \frac{\sqrt{\Lambda}}{2\sqrt{3}} \bar{\epsilon} \gamma_\mu. \quad (44)$$

It is important to realize that $\delta \Psi_\mu^2$ (resp. $\delta \Psi_\mu^4$) is consistent with $\delta \Psi_\mu^1$ (resp. $\delta \Psi_\mu^3$) in (27) when applying the $N = 1$ condition (41) with the matrices (40), which will no longer be the case for $D = 4$ pure de Sitter supergravity as shown in the next section.

6 $D = 4$ pure de Sitter supergravity

In this section the minus sign of the cosmological constant term (21) is considered. The following 4x4 matrices satisfy the conditions (37) when it is a minus sign in the first condition:

$$\begin{aligned} \mathbf{S}_1 &= \begin{pmatrix} S_1 & 0 \\ 0 & S_1 \end{pmatrix} \text{ with } S_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \mathbf{M}_1 &= \begin{pmatrix} M_1 & 0 \\ 0 & M_1 \end{pmatrix} \text{ with } M_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ \mathbf{S}_2 &= \mathbf{S}_1 \mathbf{M}_1 = \begin{pmatrix} S_2 & 0 \\ 0 & S_2 \end{pmatrix} \text{ with } S_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \quad (45)$$

¹⁸See Sec. 2.5 of [3].

Without introducing any other field than those of the graviton and the gravitino, we have thus proven that the action (19)-(23) with the matrices (45) is a locally supersymmetric action for $D = 4$ pure de Sitter supergravity.

In this case, the $N = 1$ condition (41) cannot be imposed at the beginning of the supersymmetrization process because it would lead to inconsistencies between $\delta\Psi_\mu^2$ (resp. $\delta\Psi_\mu^4$) and $\delta\Psi_\mu^1$ (resp. $\delta\Psi_\mu^3$) in (27) with the matrices (45). Without surprise, this shows once again that $D = 4, N = 1$ pure de Sitter supergravity cannot be directly obtained from a locally supersymmetric action.

Imposing the $N = 1$ condition (41) to the action (19)-(23) with the matrices (45) leads therefore to supersymmetry breaking and gives the following action for $D = 4, N = 1$ pure de Sitter supergravity:

$$S = k_1 \int dx^4 e e_m^\mu e_n^\nu R_{\mu\nu}^{mn} - 2k_1 \int dx^4 e \Lambda - k_2 \int dx^4 e (\bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho), \quad (46)$$

with

$$\begin{aligned} R_{\mu\nu}^{mn} &\equiv \partial_\mu \omega_\nu^{mn} - \partial_\nu \omega_\mu^{mn} + \omega_\mu^m \omega_\nu^{rn} - \omega_\nu^m \omega_\mu^{rn}, \\ D_\mu \psi_\nu &\equiv \partial_\mu \psi_\nu + \frac{1}{4} \omega_{\mu mn} \gamma^{mn} \psi_\nu \implies D_\mu \bar{\psi}_\nu = \partial_\mu \bar{\psi}_\nu - \frac{1}{4} \omega_{\mu mn} \bar{\psi}_\nu \gamma^{mn}, \\ D_{[\rho} e_{\sigma]}^s &= \frac{k_2}{4k_1} \lambda (\bar{\psi}_\rho \gamma^s \psi_\sigma), \\ \omega_\mu^{mn} &= \frac{1}{2} e^{m\rho} (\partial_\mu e_\rho^n - \partial_\rho e_\mu^n) - \frac{1}{2} e^{n\rho} (\partial_\mu e_\rho^m - \partial_\rho e_\mu^m) - \frac{1}{2} e^{m\rho} e^{n\sigma} (\partial_\rho e_\sigma^r - \partial_\sigma e_\rho^r) e_{r\mu} + K_\mu^{mn}(\psi), \\ K_\mu^{mn}(\psi) &= \frac{k_2}{4k_1} \lambda (\bar{\psi}_\mu \gamma^m \psi^n - \bar{\psi}_\mu \gamma^n \psi^m + \bar{\psi}^m \gamma_\mu \psi^n). \end{aligned}$$

It is worth noting that the action (46) has no mass term because (23) vanishes when imposing the $N = 1$ condition (41) with the matrices (45), which differs from the action also obtained by supersymmetry breaking in Section 4.3 of [6].

7 Discussion

7.1 Superalgebras

The heterodox approach used in this paper to derive $D = 4$ pure de Sitter supergravity considers that $N = 2$ during the supersymmetrization process. It is known that there exist de Sitter superalgebras for even N but they do not allow unitary representations¹⁹. Therefore, superalgebras do not play an important role in pure de Sitter supergravity.

7.2 Killing spinors

The Killing spinor analysis is similar to the one given in Sec. 2.2.3 of [9]. When $\Psi_\mu^i = 0$ the bosonic field equations derived from the action (19)-(23) are

$$e_n^\nu R_{\mu\nu}^{mn} - \frac{1}{2} e_\mu^m R \mp e_\mu^m \Lambda = 0 \quad (47)$$

¹⁹See footnote 1 of [6] and [7, 8].

whose homogeneous solution is (anti-)de Sitter space with the curvature tensor

$$R_{\mu\nu}^{mn} = \mp \frac{\Lambda}{3} (e_\mu^m e_\nu^n - e_\nu^m e_\mu^n). \quad (48)$$

The conditions $\Psi_\mu^i = 0 \Rightarrow \delta\Psi_\mu^i = 0$ lead from (27) to the Killing spinor equation

$$\tilde{D}_\mu \varepsilon^i \equiv D_\mu \varepsilon^i + \frac{\sqrt{\Lambda}}{2\sqrt{3}} M_1^{ij} \gamma_\mu \varepsilon^j = 0, \quad (49)$$

whose integrability condition is²⁰

$$[\tilde{D}_\mu, \tilde{D}_\nu] \varepsilon^i = \left(\frac{1}{4} R_{\mu\nu}^{mn} \gamma_{mn} \pm \frac{\Lambda}{6} \gamma_{\mu\nu} \right) \varepsilon^i = 0. \quad (50)$$

Substituting (48) into (50), one can see that the integrability condition is obeyed and therefore that Killing spinors do exist even in the de Sitter case, which differs from Section 4.2 of [6].

7.3 No extended pure de Sitter supergravities

It is known that the actions for extended supergravities contain at least one term that is based on the spinor bilinear²¹ $\bar{\Psi}_\mu^i \gamma^{[\mu} \gamma_{\rho\sigma} \gamma^{\nu]} \Psi_\nu^j$ that would vanish from its anti-symmetry in i, j by construction of (17). Extended purely pure de Sitter supergravities are therefore not allowed. *This should not be considered a problem because it is known that the other forces of physics are treated differently.*

8 Conclusion

It is possible to derive $D = 4$ pure de Sitter supergravity from a locally supersymmetric action without having to introduce any other field than those of the graviton and the gravitino. The price to pay is to double the usual number of Majorana spinors for both the gravitino field and the supersymmetry parameter, which allows to construct spinor-ansatzes that cancel the usual quartic fermion terms appearing in the supersymmetrization process. The action for $D = 4, N = 1$ pure de Sitter supergravity can be obtained through supersymmetry breaking and it is worth noting that it has no mass term. One direct consequence of this heterodox approach is that extended pure de Sitter supergravities are not allowed.

Appendix

A Representations of the γ -matrices

The four most used representations of the γ -matrices are given in this Appendix, starting with the Majorana real representation. The corresponding expression for a Majorana spinor $\chi_M^C \equiv i\gamma^0 C^\dagger \chi_M^* = \chi_M$ is given, based on 4 real anti-commuting Grassmann variables $\alpha, \beta, \gamma, \delta$.

It is known that two sets of γ -matrices are related by

$$\gamma'^m = U \gamma^m U^\dagger,$$

²⁰This result is calculated in Appendix E.2.

²¹See for instance Sec. 2.8 of [3].

$$C' = U^* C U^\dagger, \\ \chi' = U \chi,$$

where U is a unitary matrix: $U^\dagger = U^{-1}$. The expression for the matrix U allowing to obtain the considered representation of the γ -matrices from the Majorana real one is also given.

Majorana real representation

$$\gamma^0 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\gamma_* = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \chi_M = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}$$

Another real representation

$$\gamma^0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\gamma_* = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & 1 \end{pmatrix} \Rightarrow \chi_M = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha + \gamma \\ -\alpha + \gamma \\ -\beta - \delta \\ -\beta + \delta \end{pmatrix}$$

Weyl representation

$$\gamma^0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\gamma_* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, C = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, U = \frac{1}{2} \begin{pmatrix} 1 & i & 1 & -i \\ -i & 1 & i & 1 \\ -i & -1 & i & -1 \\ 1 & -i & 1 & i \end{pmatrix} \Rightarrow \chi_M = \frac{1}{2} \begin{pmatrix} (\alpha + \gamma) + i(\beta - \delta) \\ (\beta + \delta) - i(\alpha - \gamma) \\ -(\beta + \delta) - i(\alpha - \gamma) \\ (\alpha + \gamma) - i(\beta - \delta) \end{pmatrix}$$

Dirac representation

$$\gamma^0 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$\gamma_* = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, U = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & -i & 0 \\ -1 & 0 & 0 & -i \\ i & 0 & 0 & 1 \\ 0 & i & -1 & 0 \end{pmatrix} \Rightarrow \chi_M = \frac{1}{\sqrt{2}} \begin{pmatrix} \beta - i\gamma \\ -\alpha - i\delta \\ \delta + i\alpha \\ -\gamma + i\beta \end{pmatrix}$$

B consistency

B.1 Consistency of the construction for spinor-ansatzes (17)

Let's prove the expression for Ψ^3 (the proof for Ψ^4 is similar and let to the reader). Since $\Psi^1 \equiv \frac{1}{\sqrt{2}}P_L\psi^1 \equiv \frac{1}{2\sqrt{2}}(\mathbb{1} + \gamma_*)\psi^1$, we have

$$\begin{aligned}
\Psi^3 &\equiv (\Psi^1)^C, \\
&= \frac{1}{2\sqrt{2}}[(\mathbb{1} + \gamma_*)\psi^1]^C, \\
&\stackrel{(a)}{=} \frac{i}{2\sqrt{2}}\gamma^0 C^\dagger[(\mathbb{1} + \gamma_*)\psi^1]^*, \\
&= \frac{i}{2\sqrt{2}}\gamma^0 C^\dagger(\psi^1)^* + \frac{i}{2\sqrt{2}}\gamma^0 C^\dagger(\gamma_*)^*(\psi^1)^*, \\
&\stackrel{(b)}{=} \frac{i}{2\sqrt{2}}\gamma^0 C^\dagger(\psi^1)^* + \frac{i}{2\sqrt{2}}\gamma^0 \gamma_* C^\dagger(\psi^1)^*, \\
&\stackrel{(c)}{=} \frac{i}{2\sqrt{2}}\gamma^0 C^\dagger(\psi^1)^* - \frac{i}{2\sqrt{2}}\gamma_* \gamma^0 C^\dagger(\psi^1)^*, \\
&\stackrel{(d)}{=} \frac{1}{2\sqrt{2}}(\psi^1)^C - \frac{1}{2\sqrt{2}}\gamma_* (\psi^1)^C, \\
&= \frac{1}{2\sqrt{2}}(\mathbb{1} - \gamma_*)(\psi^1)^C, \\
&\stackrel{(e)}{=} \frac{1}{2\sqrt{2}}(\mathbb{1} - \gamma_*)\psi^1, \\
&\equiv \frac{1}{\sqrt{2}}P_R\psi^1.
\end{aligned}$$

The step (a) uses the definition (2).

The step (b) uses the property $C^\dagger(\gamma_*)^* = \gamma_* C^\dagger$ coming from (3) with $(\gamma_*)^\dagger = \gamma_*$.

The step (c) uses the relation $\gamma^0 \gamma^* = -\gamma^* \gamma^0$.

The step (d) uses the definition (2).

The step (e) comes from the fact that ψ^1 is a Majorana spinor.

Let's also prove the expression for $\bar{\Psi}^1$ (the proofs for $\bar{\Psi}^2$, $\bar{\Psi}^3$, $\bar{\Psi}^4$ are similar and are let to the reader). Since $\Psi^1 \equiv \frac{1}{\sqrt{2}}P_L\psi^1 \equiv \frac{1}{2\sqrt{2}}(\mathbb{1} + \gamma_*)\psi^1$, we have

$$\begin{aligned}
\bar{\Psi}^1 &\stackrel{(a)}{=} (\Psi^1)^T C, \\
&= \frac{1}{2\sqrt{2}}[(\mathbb{1} + \gamma_*)\psi^1]^T C, \\
&= \frac{1}{2\sqrt{2}}(\psi^1)^T C + \frac{1}{2\sqrt{2}}(\psi^1)^T (\gamma_*)^T C, \\
&\stackrel{(b)}{=} \frac{1}{2\sqrt{2}}(\psi^1)^T C + \frac{1}{2\sqrt{2}}(\psi^1)^T C \gamma_*, \\
&= (\psi^1)^T C \frac{1}{2\sqrt{2}}(\mathbb{1} + \gamma_*),
\end{aligned}$$

$$\begin{aligned}
&\stackrel{(c)}{=} \bar{\psi}^1 \frac{1}{2\sqrt{2}} (\mathbb{1} + \gamma_*) , \\
&\equiv \frac{1}{\sqrt{2}} P_L \bar{\psi}^1 .
\end{aligned} \tag{51}$$

The step (a) uses the definition (5).

The step (b) uses the property $(\gamma_*)^T C = C \gamma_*$ coming from (3).

The step (c) uses the definition (5).

B.2 Consistency of the local supersymmetry transformation (27)

Let's prove the expression for $\delta\Psi_\mu^4$ (the proof for $\delta\Psi_\mu^3$ is similar and left to the reader). Taking into account (13),(27),(40),(45) we have $\delta\Psi_\mu^2 = D_\mu \epsilon^2 + k_4 \sqrt{\Lambda} M_1^{2h} \gamma_\mu \epsilon^h = \partial_\mu \epsilon^2 + \frac{1}{4} \omega_{\mu ab} \gamma^{ab} \epsilon^2 \pm k_4 \sqrt{\Lambda} \gamma_\mu \epsilon^1$. Hence

$$\begin{aligned}
\delta\Psi_\mu^4 &\equiv (\delta\Psi_\mu^2)^C , \\
&= (\partial_\mu \epsilon^2 + \frac{1}{4} \omega_{\mu ab} \gamma^{ab} \epsilon^2 \pm k_4 \sqrt{\Lambda} \gamma_\mu \epsilon^1)^C , \\
&\stackrel{(a)}{=} B(\partial_\mu \epsilon^2 + \frac{1}{4} \omega_{\mu ab} \gamma^{ab} \epsilon^2 \pm k_4 \sqrt{\Lambda} \gamma_\mu \epsilon^1)^* , \\
&= B(\partial_\mu \epsilon^2)^* + \frac{1}{4} B(\omega_{\mu ab} \gamma^{ab} \epsilon^2)^* \pm B(k_4 \sqrt{\Lambda} \gamma_\mu \epsilon^1)^* , \\
&= B\partial_\mu(\epsilon^2)^* + \frac{1}{4} B\omega_{\mu ab}(\gamma^{ab})^*(\epsilon^2)^* \pm Bk_4 \sqrt{\Lambda}(\gamma_\mu)^*(\epsilon^1)^* , \\
&= \partial_\mu[B(\epsilon^2)^*] + \frac{1}{4} \omega_{\mu ab} B(\gamma^{ab})^* B^{-1} B(\epsilon^2)^* \pm k_4 \sqrt{\Lambda} B(\gamma_\mu)^* B^{-1} B(\epsilon^1)^* , \\
&\stackrel{(b)}{=} \partial_\mu[B(\epsilon^2)^*] + \frac{1}{4} \omega_{\mu ab} \gamma^{ab} B(\epsilon^2)^* \pm k_4 \sqrt{\Lambda} \gamma_\mu B(\epsilon^1)^* , \\
&\stackrel{(c)}{=} \partial_\mu \epsilon^4 + \frac{1}{4} \omega_{\mu ab} \gamma^{ab} \epsilon^4 \pm k_4 \sqrt{\Lambda} \gamma_\mu \epsilon^3 , \\
&\stackrel{!}{=} D_\mu \epsilon^4 + k_4 \sqrt{\Lambda} M_1^{4h} \gamma_\mu \epsilon^h .
\end{aligned}$$

The step (a) uses the definition (2) with $B = i\gamma^0 C^\dagger$.

The step (b) uses the property²² $BN^*B^{-1} = N$ where N is any matrix obtained from products of γ^m and $i\gamma_*$.

The step (c) uses the definition (2) with $B = i\gamma^0 C^\dagger$.

B.3 Consistency of the Lorentz transformation of the supersymmetry transformations (26)-(27)

Let's prove that (26) transforms by (8) as a vector, that is $\delta_L(\delta e_\mu^m) \equiv -\lambda^m{}_k \delta e_\mu^k$

$$\begin{aligned}
\delta_L(\delta e_\mu^m) &= \delta_L[k_3 S_1^{ij}(\bar{\epsilon}^i \gamma^m \Psi_\mu^j)] , \\
&= k_3 S_1^{ij}(\delta_L \bar{\epsilon}^i \gamma^m \Psi_\mu^j) + k_3 S_1^{ij}(\bar{\epsilon}^i \gamma^m \delta_L \Psi_\mu^j) ,
\end{aligned}$$

²²See Appendix C for further details.

$$\begin{aligned}
&\stackrel{(a)}{=} \frac{k_3}{4} \lambda_{ab} \mathbf{S}_1^{ij} (\bar{\varepsilon}^i \gamma^{ab} \gamma^m \Psi_\mu^j) - \frac{k_3}{4} \lambda_{ab} \mathbf{S}_1^{ij} (\bar{\varepsilon}^i \gamma^m \gamma^{ab} \Psi_\mu^j), \\
&\stackrel{(b)}{=} \frac{k_3}{4} \lambda_{ab} \mathbf{S}_1^{ij} (\bar{\varepsilon}^i \gamma^{ab} \gamma^m \Psi_\mu^j) - \frac{k_3}{4} \lambda_{ab} \mathbf{S}_1^{ij} (\bar{\varepsilon}^i \gamma^m \gamma^{ab} \Psi_\mu^j), \\
&= -\lambda^m{}_k k_3 \mathbf{S}_1^{ij} (\bar{\varepsilon}^i \gamma^k \Psi_\mu^j), \\
&\stackrel{!}{=} -\lambda^m{}_k \delta e_\mu^k,
\end{aligned}$$

The step (a) uses (27)

The step (b) uses the relation $\gamma^{ab} \gamma^m - \gamma^m \gamma^{ab} = 2\eta^{bm} \gamma^a - 2\eta^{am} \gamma^b$

Let's prove that (27) transforms by (9) as a spinor, that is $\delta_L(\delta\Psi_\mu^i) \equiv -\frac{1}{4} \lambda_{ab} \gamma^{ab} \delta\Psi_\mu^i$

$$\begin{aligned}
\delta_L(\delta\Psi_\mu^i) &= \delta_L(D_\mu \varepsilon^i + k_4 \sqrt{\Lambda} \mathbf{M}_1^{ih} \gamma_\mu \varepsilon^h), \\
&\stackrel{(a)}{=} \delta_L(\partial_\mu \varepsilon^i + \frac{1}{4} \omega_{\mu ab} \gamma^{ab} \varepsilon^i + k_4 \sqrt{\Lambda} e_\mu^m \mathbf{M}_1^{ih} \gamma_m \varepsilon^h), \\
&= \partial_\mu \delta_L \varepsilon^i + \frac{1}{4} \delta_L \omega_{\mu ab} \gamma^{ab} \varepsilon^i + \frac{1}{4} \omega_{\mu ab} \gamma^{ab} \delta_L \varepsilon^i + k_4 \sqrt{\Lambda} \delta_L e_\mu^m \mathbf{M}_1^{ih} \gamma_m \varepsilon^h + k_4 \sqrt{\Lambda} e_\mu^m \mathbf{M}_1^{ih} \gamma_m \delta_L \varepsilon^h, \\
&\stackrel{(b)}{=} \partial_\mu \left(-\frac{1}{4} \lambda_{ab} \gamma^{ab} \varepsilon^i \right) + \frac{1}{4} (\partial_\mu \lambda_{ab} - \lambda_a{}^k \omega_{\mu kb} + \lambda_b{}^k \omega_{\mu ka}) \gamma^{ab} \varepsilon^i + \frac{1}{4} \omega_{\mu ab} \gamma^{ab} \left(-\frac{1}{4} \lambda_{cd} \gamma^{cd} \varepsilon^i \right) \\
&\quad + k_4 \sqrt{\Lambda} (-\lambda^m{}_k e_\mu^k) \mathbf{M}_1^{ih} \gamma_m \varepsilon^h + k_4 \sqrt{\Lambda} e_\mu^m \mathbf{M}_1^{ih} \gamma_m \left(-\frac{1}{4} \lambda_{ab} \gamma^{ab} \varepsilon^h \right), \\
&= -\frac{1}{4} \cancel{\partial_\mu \lambda_{ab}} \gamma^{ab} \overset{a}{\varepsilon^i} - \frac{1}{4} \lambda_{ab} \gamma^{ab} \cancel{\partial_\mu \varepsilon^i} + \frac{1}{4} \cancel{\partial_\mu \lambda_{ab}} \gamma^{ab} \overset{a}{\varepsilon^i} - \frac{1}{4} \lambda_a{}^k \omega_{\mu kb} \gamma^{ab} \varepsilon^i + \frac{1}{4} \lambda_b{}^k \omega_{\mu ka} \gamma^{ab} \varepsilon^i \\
&\quad - \frac{1}{16} \lambda_{ab} \omega_{\mu cd} \gamma^{cd} \gamma^{ab} \varepsilon^i - \lambda^m{}_k k_4 \sqrt{\Lambda} e_\mu^k \mathbf{M}_1^{ih} \gamma_m \varepsilon^h - \frac{1}{4} \lambda_{ab} k_4 \sqrt{\Lambda} e_\mu^m \mathbf{M}_1^{ih} \gamma_m \gamma^{ab} \varepsilon^h, \\
&\stackrel{(c)}{=} -\frac{1}{4} \lambda_{ab} \gamma^{ab} \cancel{\partial_\mu \varepsilon^i} - \frac{1}{4} \lambda_a{}^k \omega_{\mu kb} \gamma^{ab} \overset{a}{\varepsilon^i} + \frac{1}{4} \lambda_b{}^k \omega_{\mu ka} \gamma^{ab} \overset{a}{\varepsilon^i} - \frac{1}{16} \lambda_{ab} \gamma^{ab} \omega_{\mu cd} \gamma^{cd} \varepsilon^i \\
&\quad + \frac{1}{4} \lambda_a{}^k \omega_{\mu kb} \gamma^{ab} \overset{a}{\varepsilon^i} - \frac{1}{4} \lambda_b{}^k \omega_{\mu ka} \gamma^{ab} \overset{b}{\varepsilon^i} - \lambda^m{}_k k_4 \sqrt{\Lambda} e_\mu^k \mathbf{M}_1^{ih} \gamma_m \overset{c}{\varepsilon^h} - \frac{1}{4} \lambda_{ab} \gamma^{ab} k_4 \sqrt{\Lambda} \mathbf{M}_1^{ih} \gamma_m \varepsilon^h \\
&\quad + \lambda^m{}_k k_4 \sqrt{\Lambda} e_\mu^k \mathbf{M}_1^{ih} \gamma_m \varepsilon^h, \\
&= -\frac{1}{4} \lambda_{ab} \gamma^{ab} (\partial_\mu \varepsilon^i + \frac{1}{4} \omega_{\mu cd} \gamma^{cd} \varepsilon^i + k_4 \sqrt{\Lambda} \mathbf{M}_1^{ih} \gamma_m \varepsilon^h), \\
&\stackrel{(d)}{=} -\frac{1}{4} \lambda_{ab} \gamma^{ab} (D_\mu \varepsilon^i + k_4 \sqrt{\Lambda} \mathbf{M}_1^{ih} \gamma_m \varepsilon^h), \\
&\stackrel{!}{=} -\frac{1}{4} \lambda_{ab} \gamma^{ab} \delta \Psi_\mu^i,
\end{aligned}$$

The step (a) uses (13).

The step (b) uses (8)-(9) and the result obtained in Appendix E.3 for the Lorentz transformation of the spin connection $\omega_\mu{}^{mn}$.

The step (c) uses the relation $\gamma^{cd} \gamma^{ab} = \gamma^{ab} \gamma^{cd} + 2\eta^{ac} \gamma^{bd} - 2\eta^{bc} \gamma^{ad} - 2\eta^{ad} \gamma^{bc} + 2\eta^{bd} \gamma^{ac}$ and $\gamma_m \gamma^{ab} = \gamma^{ab} \gamma_m - 2\eta_m^b \gamma^a + 2\eta_m^a \gamma^b$. The step (c) also uses the properties $\omega_{\mu ab} = -\omega_{\mu ba}$ and $\lambda_{ab} = -\lambda_{ba}$.

The step (d) uses (13).

C Proof that spinor-ansatz bilinears are real

Let's consider the spinor-ansatz bilinear $\mathbf{M}^{ij}\bar{\epsilon}^i N \Psi^j$ based on any two spinor-ansatzes ϵ^i, Ψ^i defined by (17) and where N is any matrix obtained from products of γ^m and $i\gamma_*$. For any spinor χ it is easy to show from (3),(6) that $\chi^C = B\chi^*$ and $\bar{\chi}^C = -(\bar{\chi})^*B^{-1}$ where $B = i\gamma^0 C^\dagger \Rightarrow B^{-1} = iC\gamma^0$. From the properties (3) and $(\gamma^m)^\dagger = \gamma^0 \gamma^m \gamma^0$ with $(\gamma^0)^2 = -1$, it can also be shown that $BN^*B^{-1} = N$. Hence²³

$$\begin{aligned} (\mathbf{M}^{ij}\bar{\epsilon}^i N \Psi^j)^* &= -\mathbf{M}^{ij}(\bar{\epsilon}^i)^* N^*(\Psi^j)^*, \\ &= -\mathbf{M}^{ij}(\bar{\epsilon}^i)^* B^{-1} BN^* B^{-1} B(\Psi^j)^*, \\ &= \mathbf{M}^{ij}\overline{(\bar{\epsilon}^i)^C} N(\Psi^j)^C, \\ &= \mathbf{M}^{11}\overline{(\bar{\epsilon}^1)^C} N(\Psi^1)^C + \mathbf{M}^{12}\overline{(\bar{\epsilon}^1)^C} N(\Psi^2)^C + \mathbf{M}^{21}\overline{(\bar{\epsilon}^2)^C} N(\Psi^1)^C + \mathbf{M}^{22}\overline{(\bar{\epsilon}^2)^C} N(\Psi^2)^C \\ &\quad + \mathbf{M}^{33}\overline{(\bar{\epsilon}^3)^C} N(\Psi^3)^C + \mathbf{M}^{34}\overline{(\bar{\epsilon}^3)^C} N(\Psi^4)^C + \mathbf{M}^{43}\overline{(\bar{\epsilon}^4)^C} N(\Psi^3)^C + \mathbf{M}^{44}\overline{(\bar{\epsilon}^4)^C} N(\Psi^4)^C, \\ &= \mathbf{M}^{11}\bar{\epsilon}^3 N \Psi^3 + \mathbf{M}^{12}\bar{\epsilon}^3 N \Psi^4 + \mathbf{M}^{21}\bar{\epsilon}^4 N \Psi^3 + \mathbf{M}^{22}\bar{\epsilon}^4 N \Psi^4 \\ &\quad + \mathbf{M}^{33}\bar{\epsilon}^1 N \Psi^1 + \mathbf{M}^{34}\bar{\epsilon}^1 N \Psi^2 + \mathbf{M}^{43}\bar{\epsilon}^2 N \Psi^1 + \mathbf{M}^{44}\bar{\epsilon}^2 N \Psi^2, \\ &\stackrel{!}{=} \mathbf{M}^{ij}\bar{\epsilon}^i N \Psi^j, \end{aligned}$$

since $\mathbf{M}^{11} = \mathbf{M}^{33}$, $\mathbf{M}^{12} = \mathbf{M}^{34}$, $\mathbf{M}^{21} = \mathbf{M}^{43}$, $\mathbf{M}^{22} = \mathbf{M}^{44}$ by construction.

D Detailed calculations

The calculations are cumbersome but not complicated. They require to master Levi-Civita transformations, spinor-ansatz flips, partial integration, supersymmetry transformations and spinor-ansatz reorderings.

Levi-Civita transformations simplify the calculations because they are in most cases easier to perform on expressions that are based on the totally antisymmetric Levi-Civita symbol $\epsilon^{\mu\nu\rho\sigma} \equiv e \epsilon^{mnrs} e_m^\mu e_n^\nu e_r^\rho e_s^\sigma \Rightarrow \epsilon_{\mu\nu\rho\sigma} = e^{-1} \epsilon_{mnrs} e_\mu^m e_\nu^n e_\rho^r e_\sigma^s$ with $\epsilon_{0123} = +1 = -\epsilon^{0123}$ and $e = \det e_\mu^m$. To do so, the following relations are needed

$$\begin{aligned} e\mathbb{1} &= -\frac{i}{24} \epsilon^{\mu\nu\rho\sigma} \gamma_* \gamma_{\mu\nu\rho\sigma} \iff i\gamma_* = \frac{1}{24} \epsilon_{mnrs} \gamma^{mnrs}, \\ e\gamma^\mu &= \frac{i}{6} \epsilon^{\mu\nu\rho\sigma} \gamma_* \gamma_{\nu\rho\sigma} \iff i\gamma_* \gamma_m = -\frac{1}{6} \epsilon_{mnrs} \gamma^{nrs}, \\ e\gamma^{\mu\nu} &= \frac{i}{2} \epsilon^{\mu\nu\rho\sigma} \gamma_* \gamma_{\rho\sigma} \iff i\gamma_* \gamma_{mn} = -\frac{1}{2} \epsilon_{mnrs} \gamma^{rs}, \\ e\gamma^{\mu\nu\rho} &= -i\epsilon^{\mu\nu\rho\sigma} \gamma_* \gamma_\sigma \iff i\gamma_* \gamma_{mnr} = \epsilon_{mnrs} \gamma^s, \\ e\gamma^{\mu\nu\rho\sigma} &= -i\epsilon^{\mu\nu\rho\sigma} \gamma_* \iff i\gamma_* \gamma_{mnrs} = \epsilon_{mnrs} \mathbb{1}, \end{aligned}$$

where $\mathbb{1}$ is the identity matrix in the spinor space.

Spinor-ansatz flips are performed by taking the transpose of spinor-ansatz bilinears, by taking into account the properties (3) and by incorporating a minus sign obtained by changing

²³As in Sec. 3.2.4 of [4], the convention is used to reverse the order of anti-commuting Grassmann variables (spinor components) in the process of complex conjugation: $(\alpha\beta)^* = \beta^*\alpha^* = -\alpha^*\beta^*$ where α, β are two anti-commuting Grassmann variables.

the order of the spinor components that are anti-commuting Grassmann variables. Let's give an example:

$$\begin{aligned}
(\bar{\Psi}_\rho^i \gamma_* \gamma_n \Psi_\sigma^j) &= (\bar{\Psi}_\rho^i \gamma_* \gamma_n \Psi_\sigma^j)^T, \\
&\stackrel{(a)}{=} [(\Psi_\rho^i)^T C \gamma_* \gamma_n \Psi_\sigma^j]^T, \\
&\stackrel{(b)}{=} -[(\Psi_\sigma^j)^T (\gamma_n)^T (\gamma_*)^T C^T \Psi_\mu^i], \\
&\stackrel{(c)}{=} -[(\Psi_\sigma^j)^T (\gamma_n)^T C^T \gamma_* \Psi_\mu^i], \\
&\stackrel{(d)}{=} -[(\Psi_\sigma^j)^T C \gamma_n \gamma_* \Psi_\mu^i], \\
&\stackrel{(e)}{=} [(\Psi_\sigma^j)^T C \gamma_* \gamma_n \Psi_\mu^i], \\
&\stackrel{(f)}{=} (\bar{\Psi}_\sigma^j \gamma_* \gamma_n \Psi_\rho^i).
\end{aligned}$$

The step (a) uses the definition (5).

The step (b) incorporates a minus sign obtained by changing the order of the spinor component that are anti-commuting Grassmann variables

The step (c) uses the property $(\gamma_*)^T C^T = C^T \gamma_*$ coming from (3).

The step (d) uses the property $(\gamma_n)^T C^T = C^T \gamma_n$ coming from (3).

The step (e) uses the relation $\gamma_n \gamma_* = -\gamma_* \gamma_n$.

The step (f) uses the definition (5).

Partial integration is the cornerstone of the least action principle. It is performed on the local Lorentz derivative D_μ by taking into account that the local Lorentz derivative of a scalar is its partial derivative and that the local Lorentz derivative acts distributively: $\partial_\mu (S_2^{ij} \bar{\Psi}_\nu^i \gamma^{\nu\rho} \Psi_\rho^j) = D_\mu (S_2^{ij} \bar{\Psi}_\nu^i \gamma^{\nu\rho} \Psi_\rho^j) = S_2^{ij} D_\mu \bar{\Psi}_\nu^i \gamma^{\nu\rho} \Psi_\rho^j + S_2^{ij} \bar{\Psi}_\nu^i D_\mu \gamma^{\nu\rho} \Psi_\rho^j + S_2^{ij} \bar{\Psi}_\nu^i \gamma^{\nu\rho} D_\mu \Psi_\rho^j$. Moreover, since ϵ_{mnrs} , $\epsilon^{\mu\nu\rho\sigma}$ are numbers²⁴, one has $D_\mu \epsilon_{mnrs} = 0$ and $D_\mu \epsilon^{\nu\rho\sigma\alpha} = 0$. Finally, since $D_\mu \gamma^m = 0$ and $D_\mu \gamma_* = 0$ from (14)-(15), one has $D_\mu \gamma^\nu = \gamma^m D_\mu e_m^\nu$ and $D_\mu (\gamma_* \gamma^\nu) = \gamma_* \gamma^m D_\mu e_m^\nu$.

Supersymmetry transformations are performed by applying (26)-(27) when needed in the course of the calculations.

Spinor-ansatz reorderings allow the interchange of spinor-ansatzes between two spinor-ansatz bilinears. They are performed using the fundamental Fierz rearrangement identity

$$\begin{aligned}
(\bar{\chi}_1 \chi_2)(\bar{\chi}_3 \chi_4) &= -\frac{1}{4}(\bar{\chi}_3 \chi_2)(\bar{\chi}_1 \chi_4) - \frac{1}{4}(\bar{\chi}_3 \gamma_* \chi_2)(\bar{\chi}_1 \gamma_* \chi_4) \\
&\quad -\frac{1}{4}(\bar{\chi}_3 \gamma_a \chi_2)(\bar{\chi}_1 \gamma^a \chi_4) + \frac{1}{4}(\bar{\chi}_3 \gamma_* \gamma_a \chi_2)(\bar{\chi}_1 \gamma_* \gamma^a \chi_4) \\
&\quad +\frac{1}{8}(\bar{\chi}_3 \gamma_{ab} \chi_2)(\bar{\chi}_1 \gamma^{ab} \chi_4),
\end{aligned}$$

where χ_1 , χ_2 , χ_3 , and χ_4 are any four spinors.

These five operations are respectively indicated by L, F, P, S, R placed above the equal sign in the course of the calculations.

²⁴See Sec. 7.5 of [4].

The calculations also require specific relations coming from the algebra of γ -matrices that follow from their defining relation $\gamma^m \gamma^n + \gamma^n \gamma^m = 2\eta^{mn} \mathbb{1}$. These relations are specified when needed in the course of the calculations.

Finally, changes of indices are performed throughout the calculations on Lorentz (latin) indices, spacetime (greek) indices and the matrix indices. These changes of indices are allowed because the calculations are performed on expressions with contracted indices.

D.1 Derivation with respect to the spin connection $\omega_\mu{}^{mn}$

From (20) we have

$$\begin{aligned}
\delta_\omega S_{\text{EH}} &= +k_1 \delta_\omega \int dx^4 e e_m^\mu e_n^\nu R_{\mu\nu}{}^{mn}, \\
&\stackrel{(a)}{=} +\frac{k_1}{2} \delta_\omega \int dx^4 e (e_m^\mu e_n^\nu - e_n^\mu e_m^\nu) R_{\mu\nu}{}^{mn}, \\
&\stackrel{(b)}{=} -\frac{k_1}{4} \delta_\omega \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnrs} R_{\mu\nu}{}^{mn} e_\rho^r e_\sigma^s, \\
&\stackrel{(c)}{=} -\frac{k_1}{4} \delta_\omega \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnrs} (\partial_\mu \omega_\nu{}^{mn} - \partial_\nu \omega_\mu{}^{mn} + \omega_\mu{}^m{}_k \omega_\nu{}^{kn} - \omega_\nu{}^m{}_k \omega_\mu{}^{kn}) e_\rho^r e_\sigma^s, \\
&= -\frac{k_1}{2} \delta_\omega \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnrs} (\partial_\mu \omega_\nu{}^{mn} + \omega_\mu{}^m{}_k \omega_\nu{}^{kn}) e_\rho^r e_\sigma^s, \\
&= -\frac{k_1}{2} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnrs} (\partial_\mu \delta \omega_\nu{}^{mn} + \omega_\mu{}^m{}_k \delta \omega_\nu{}^{kn} + \omega_\mu{}^n{}_k \delta \omega_\nu{}^{mk}) e_\rho^r e_\sigma^s, \\
&\stackrel{(d)}{=} -\frac{k_1}{2} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnrs} D_\mu \delta \omega_\nu{}^{mn} e_\rho^r e_\sigma^s, \\
&\stackrel{P}{=} +\frac{k_1}{2} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnrs} \delta \omega_\nu{}^{mn} D_\mu e_\rho^r e_\sigma^s + \frac{k_1}{2} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnrs} \delta \omega_\nu{}^{mn} e_\rho^r D_\mu e_\sigma^s, \\
&= +k_1 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnrs} \delta \omega_\mu{}^{mn} e_\nu^r D_\rho e_\sigma^s.
\end{aligned}$$

The step (a) uses the property $R_{\mu\nu}{}^{mn} = -R_{\mu\nu}{}^{nm}$ of (24).

The step (b) uses the relation $e (e_m^\mu e_n^\nu - e_n^\mu e_m^\nu) = -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnrs} e_\rho^r e_\sigma^s$.

The step (c) uses (24).

The step (d) uses the definition of the local Lorentz derivative of the tensor²⁵ $\delta \omega_\nu{}^{mn}$ given by $D_\mu \delta \omega_\nu{}^{mn} \equiv \partial_\mu \delta \omega_\nu{}^{mn} + \omega_\mu{}^m{}_k \delta \omega_\nu{}^{kn} + \omega_\mu{}^n{}_k \delta \omega_\nu{}^{mk}$.

From (22) we have

$$\begin{aligned}
\delta_\omega S_{\text{RS}} &= -k_2 \delta_\omega \int dx^4 e \mathbf{S}_1^{ij} (\bar{\Psi}_\mu^i \gamma^{\mu\nu\rho} D_\nu \Psi_\rho^j), \\
&\stackrel{L}{=} +ik_2 \delta_\omega \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\Psi}_\mu^i \gamma_* \gamma_\sigma D_\nu \Psi_\rho^j), \\
&\stackrel{(a)}{=} +ik_2 \delta_\omega \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} [\bar{\Psi}_\mu^i \gamma_* \gamma_\sigma (\partial_\nu \Psi_\rho^j + \frac{1}{4} \omega_{\nu mn} \gamma^{mn} \Psi_\rho^j)],
\end{aligned}$$

²⁵Note that $\delta \omega_\nu{}^{mn}$ is a tensor despite the fact that $\omega_\nu{}^{mn}$ is not.

$$\begin{aligned}
&= +\frac{ik_2}{4} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \delta\omega_{\nu mn} S_1^{ij} (\bar{\Psi}_\mu^i \gamma_* \gamma_\sigma \gamma^{mn} \Psi_\rho^j), \\
&\stackrel{C}{=} -\frac{ik_2}{4} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \delta\omega_\mu{}^{mn} S_1^{ij} (\bar{\Psi}_\rho^i \gamma_* \gamma_\nu \gamma_{mn} \Psi_\sigma^j), \\
&\stackrel{(b)}{=} -\frac{ik_2}{4} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \delta\omega_\mu{}^{mn} S_1^{ij} (\bar{\Psi}_\rho^i \gamma_* \gamma_\nu \gamma_{mn} \Psi_\sigma^j) - \frac{ik_2}{2} \int \overbrace{dx^4 \delta\omega_\mu{}^{mn} e_{m\nu} \epsilon^{\mu\nu\rho\sigma} S_1^{ij} (\bar{\Psi}_\rho^i \gamma_* \gamma_n \Psi_\sigma^j)}, \\
&\stackrel{LC}{=} -\frac{k_2}{4} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnrs} \delta\omega_\mu{}^{mn} e_\nu^r S_1^{ij} (\bar{\Psi}_\rho^i \gamma^s \Psi_\sigma^j).
\end{aligned}$$

The step (a) uses (25).

The step (b) uses the relation $\gamma_\nu \gamma_{mn} = \gamma_{\nu mn} + e_{m\nu} \gamma_n - e_{n\nu} \gamma_m$.

The expression $\epsilon^{\mu\nu\rho\sigma} S_1^{ij} (\bar{\Psi}_\rho^i \gamma_* \gamma_n \Psi_\sigma^j)$ vanishes in (b) because $S_1^{ij} (\bar{\Psi}_\rho^i \gamma_* \gamma_n \Psi_\sigma^j) \stackrel{F}{=} S_1^{ij} (\bar{\Psi}_\sigma^j \gamma_* \gamma_n \Psi_\rho^i) = S_1^{ij} (\bar{\Psi}_\sigma^j \gamma_* \gamma_n \Psi_\rho^i)$ is symmetric in $\rho\sigma$, which clashes with the total antisymmetry of $\epsilon^{\mu\nu\rho\sigma}$.

D.2 Derivation with respect to the graviton e_μ^m

From (20) we have

$$\begin{aligned}
\delta_e S_{\text{EH}} &= +k_1 \delta_e \int dx^4 e e_m^\mu e_n^\nu R_{\mu\nu}{}^{mn}, \\
&\stackrel{(a)}{=} +\frac{k_1}{2} \delta_e \int dx^4 e (e_m^\mu e_n^\nu - e_n^\mu e_m^\nu) R_{\mu\nu}{}^{mn}, \\
&\stackrel{(b)}{=} -\frac{k_1}{4} \delta_e \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnrs} R_{\mu\nu}{}^{mn} e_\rho^r e_\sigma^s, \\
&= -\frac{k_1}{4} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnrs} R_{\mu\nu}{}^{mn} \delta e_\rho^r e_\sigma^s - \frac{k_1}{4} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnrs} R_{\mu\nu}{}^{mn} e_\rho^r \delta e_\sigma^s, \\
&\stackrel{(c)}{=} -\frac{k_1}{2} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnrs} R_{\mu\nu}{}^{mn} e_\rho^r \delta e_\sigma^s, \\
&\stackrel{S}{=} -\frac{k_1 k_3}{2} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnrs} R_{\mu\nu}{}^{mn} e_\rho^r S_1^{ij} (\bar{\varepsilon}^i \gamma^s \Psi_\sigma^j).
\end{aligned}$$

The step (a) uses the properties $R_{\mu\nu}{}^{mn} = -R_{\nu\mu}{}^{mn}$ and $R_{\mu\nu}{}^{mn} = -R_{\mu\nu}{}^{nm}$ of (24).

The step (b) uses the relation $e (e_m^\mu e_n^\nu - e_n^\mu e_m^\nu) = -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnrs} e_\rho^r e_\sigma^s$.

Note that the step (c) can also be expressed in the usual form $-\frac{1}{2} \int dx^4 e (e_n^\nu R_{\mu\nu}{}^{mn} - \frac{1}{2} e_\mu^m R) \delta e_\mu^m$ of general relativity by using the relation $\epsilon^{\mu\nu\rho\sigma} \epsilon_{mnrs} e_\sigma^s = -e (e_m^\mu e_n^\nu e_r^\rho - e_n^\mu e_m^\nu e_r^\rho + e_r^\mu e_m^\nu e_n^\rho - e_r^\mu e_n^\nu e_m^\rho + e_n^\mu e_r^\nu e_m^\rho - e_m^\mu e_r^\nu e_m^\rho)$.

The step (b) uses the relation $\delta e_m^\mu = -e_n^\mu e_m^\sigma \delta e_\sigma^n$.

From (21) we have

$$\begin{aligned}
\delta_e S_\Lambda &= \pm 2k_1 \delta_e \int dx^4 e, \\
&= \pm 2k_1 \int dx^4 \delta e, \\
&\stackrel{S}{=} \pm 2k_1 k_3 \int dx^4 e S_1^{ij} (\bar{\varepsilon}^i \gamma^\mu \Psi_\mu^j).
\end{aligned}$$

From (22) we have

$$\begin{aligned}
\delta_e S_{\text{RS}} &= -k_2 \delta_e \int dx^4 e \mathbf{S}_1^{ij} (\bar{\Psi}_\mu^i \gamma^{\mu\nu\rho} D_\nu \Psi_\rho^j), \\
&\stackrel{L}{=} +ik_2 \delta_e \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\Psi}_\mu^i \gamma_* \gamma_\nu D_\rho \Psi_\sigma^j), \\
&= +ik_2 \delta_e \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\Psi}_\mu^i \gamma_* \gamma_m D_\rho \Psi_\sigma^j) e_\nu^m, \\
&= -ik_2 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\Psi}_\nu^i \gamma_* \gamma_m D_\rho \Psi_\sigma^j) \delta e_\mu^m, \\
&\stackrel{S}{=} -ik_2 k_3 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\varepsilon}^i \gamma_m \Psi_\mu^j) \mathbf{S}_1^{kl} (\bar{\Psi}_\nu^k \gamma_* \gamma^m D_\rho \Psi_\sigma^l).
\end{aligned}$$

The step (a) uses the reorderings of spinors $\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_m \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_* \gamma^m D_\rho \Psi_\sigma^l) = \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^j) - \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_* D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \Psi_\sigma^j) - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_m D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \gamma^m \Psi_\sigma^j) - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_m D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^j)$

From (23) we have

$$\begin{aligned}
\delta_e S_{g\Lambda} &= +2k_2 k_4 \sqrt{\Lambda} \delta_e \int dx^4 e \mathbf{S}_2^{ij} (\bar{\Psi}_\mu^i \gamma^{\mu\nu} \Psi_\nu^j), \\
&\stackrel{L}{=} +ik_2 k_4 \sqrt{\Lambda} \delta_e \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\bar{\Psi}_\mu^i \gamma_* \gamma_{\rho\sigma} \Psi_\nu^j), \\
&= +ik_2 k_4 \sqrt{\Lambda} \delta_e \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\bar{\Psi}_\mu^i \gamma_* \gamma_{mn} \Psi_\nu^j) e_\rho^m e_\sigma^n, \\
&= +ik_2 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\bar{\Psi}_\mu^i \gamma_* \gamma_{mn} \Psi_\nu^j) \delta e_\rho^m e_\sigma^n + ik_2 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\bar{\Psi}_\mu^i \gamma_* \gamma_{mn} \Psi_\nu^j) e_\rho^m \delta e_\sigma^n, \\
&= +2ik_2 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\bar{\Psi}_\mu^i \gamma_* \gamma_{mn} \Psi_\nu^j) e_\rho^m \delta e_\sigma^n, \\
&\stackrel{S}{=} -2ik_2 k_3 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\varepsilon}^i \gamma^m \Psi_\mu^j) \mathbf{S}_2^{kl} (\bar{\Psi}_\nu^k \gamma_* \gamma_{m\rho} \Psi_\sigma^l), \\
&\stackrel{(a)}{=} -2ik_2 k_3 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\varepsilon}^i \gamma_m \Psi_\mu^j) \mathbf{S}_2^{kl} (\bar{\Psi}_\nu^k \gamma_* \gamma^m \gamma_\rho \Psi_\sigma^l) \\
&\quad + 2ik_2 k_3 k_4 \sqrt{\Lambda} \int dx^4 \underbrace{\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\varepsilon}^i \gamma_\mu \Psi_\nu^j)}_{\mathbf{S}_1^{ij} (\bar{\varepsilon}^i \gamma_\mu \Psi_\nu^j)} \overbrace{\mathbf{S}_2^{kl} (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^l)}^{\mathbf{S}_2^{kl} (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^l)}, \\
&\stackrel{(a)}{=} -2ik_2 k_3 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\varepsilon}^i \gamma_m \Psi_\mu^j) \mathbf{S}_2^{kl} (\bar{\Psi}_\nu^k \gamma_* \gamma^m \gamma_\rho \Psi_\sigma^l).
\end{aligned}$$

The step (a) uses the relation $\gamma_{m\rho} = \gamma_m \gamma_\rho - e_{m\rho}$.

D.3 Derivation with respect to the gravitino Ψ_μ^i

From (22) we have

$$\begin{aligned}
\delta_\Psi S_{\text{RS}} &= -k_2 \delta_\Psi \int dx^4 e \mathbf{S}_1^{ij} (\bar{\Psi}_\mu^i \gamma^{\mu\nu\rho} D_\nu \Psi_\rho^j), \\
&\stackrel{L}{=} +ik_2 \delta_\Psi \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\Psi}_\mu^i \gamma_* \gamma_\nu D_\rho \Psi_\sigma^j),
\end{aligned}$$

$$\begin{aligned}
&= +ik_2 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\delta \bar{\Psi}_\mu^i \gamma_* \gamma_\nu D_\rho \Psi_\sigma^j) + ik_2 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\Psi}_\mu^i \gamma_* \gamma_\nu D_\rho \delta \Psi_\sigma^j), \\
&\stackrel{P}{=} +ik_2 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\delta \bar{\Psi}_\mu^i \gamma_* \gamma_\nu D_\rho \Psi_\sigma^j) - ik_2 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (D_\mu \bar{\Psi}_\nu^i \gamma_* \gamma_\rho \delta \Psi_\sigma^j) \\
&\quad + ik_2 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\Psi}_\mu^i \gamma_* D_\nu \gamma_\rho \delta \Psi_\sigma^j), \\
&\stackrel{F(a)}{=} +2ik_2 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\delta \bar{\Psi}_\mu^i \gamma_* \gamma_\nu D_\rho \Psi_\sigma^j) - ik_2 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\delta \bar{\Psi}_\mu^i \gamma_* D_\nu \gamma_\rho \Psi_\sigma^j), \\
&\stackrel{S}{=} +2ik_2 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} ([D_\mu \bar{\varepsilon}^i - k_4 \sqrt{\Lambda} \mathbf{M}_1^{ih} \bar{\varepsilon}^h \gamma_\mu] \gamma_* \gamma_\nu D_\rho \Psi_\sigma^j) \\
&\quad - ik_2 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} ([D_\mu \bar{\varepsilon}^i - k_4 \sqrt{\Lambda} \mathbf{M}_1^{ih} \bar{\varepsilon}^h \gamma_\mu] \gamma_* D_\nu \gamma_\rho \Psi_\sigma^j), \\
&\stackrel{(b)}{=} +2ik_2 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (D_\mu \bar{\varepsilon}^i \gamma_* \gamma_\nu D_\rho \Psi_\sigma^j) + 2ik_2 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{jh} \mathbf{M}_1^{hi} (\bar{\varepsilon}^i \gamma_* \gamma_\mu \gamma_\nu D_\rho \Psi_\sigma^j) \\
&\quad - ik_2 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (D_\mu \bar{\varepsilon}^i \gamma_* D_\nu \gamma_\rho \Psi_\sigma^j) - ik_2 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{jh} \mathbf{M}_1^{hi} (\bar{\varepsilon}^i \gamma_* \gamma_\mu D_\nu \gamma_\rho \Psi_\sigma^j), \\
&\stackrel{P}{=} -2ik_2 \int \overbrace{dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\varepsilon}^i \gamma_* D_\mu \gamma_\nu D_\rho \Psi_\sigma^j)}^a + 2ik_2 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\varepsilon}^i \gamma_* \gamma_\mu D_\nu D_\rho \Psi_\sigma^j) \\
&\quad + 2ik_2 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{jh} \mathbf{M}_1^{hi} (\bar{\varepsilon}^i \gamma_* \gamma_\mu \gamma_\nu D_\rho \Psi_\sigma^j) + ik_2 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\varepsilon}^i \gamma_* D_\mu D_\nu \gamma_\rho \Psi_\sigma^j) \\
&\quad + ik_2 \int \overbrace{dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\varepsilon}^i \gamma_* D_\mu \gamma_\nu D_\rho \Psi_\sigma^j)}^a - ik_2 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{jh} \mathbf{M}_1^{hi} (\bar{\varepsilon}^i \gamma_* \gamma_\mu D_\nu \gamma_\rho \Psi_\sigma^j), \\
&\stackrel{(c)}{=} +\frac{ik_2}{4} \int dx^4 \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu}^{mn} e_\rho^r \mathbf{S}_1^{ij} (\bar{\varepsilon}^i \gamma_* \gamma_{mnr} \Psi_\sigma^j) - ik_2 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\varepsilon}^i \gamma_* D_\mu \gamma_\nu D_\rho \Psi_\sigma^j) \\
&\quad + 2ik_2 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{jh} \mathbf{M}_1^{hi} (\bar{\varepsilon}^i \gamma_* \gamma_{\mu\nu} D_\rho \Psi_\sigma^j) + 2ik_2 k_4 \sqrt{\Lambda} \int \overbrace{dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{jh} \mathbf{M}_1^{hi} (\bar{\varepsilon}^i \gamma_* \eta_{\mu\nu} D_\rho \Psi_\sigma^j)}^a \\
&\quad - ik_2 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{jh} \mathbf{M}_1^{hi} (\bar{\varepsilon}^i \gamma_* \gamma_\mu D_\nu \gamma_\rho \Psi_\sigma^j), \\
&\stackrel{L}{=} +\frac{k_2}{4} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnrs} R_{\mu\nu}^{mn} e_\rho^r \mathbf{S}_1^{ij} (\bar{\varepsilon}^i \gamma^s \Psi_\sigma^j) - ik_2 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\varepsilon}^i \gamma_* D_\mu \gamma_\nu D_\rho \Psi_\sigma^j) \\
&\quad + 4k_2 k_4 \sqrt{\Lambda} \int dx^4 e \mathbf{S}_1^{jh} \mathbf{M}_1^{hi} (\bar{\varepsilon}^i \gamma^{\mu\nu} D_\mu \Psi_\nu^j) - ik_2 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{jh} \mathbf{M}_1^{hi} (\bar{\varepsilon}^i \gamma_* \gamma_\mu D_\nu \gamma_\rho \Psi_\sigma^j), \\
&\stackrel{(e)}{=} +\frac{k_2}{4} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnrs} R_{\mu\nu}^{mn} e_\rho^r \mathbf{S}_1^{ij} (\bar{\varepsilon}^i \gamma^s \Psi_\sigma^j) + 4k_2 k_4 \sqrt{\Lambda} \int dx^4 e \mathbf{S}_1^{jh} \mathbf{M}_1^{hi} (\bar{\varepsilon}^i \gamma^{\mu\nu} D_\mu \Psi_\nu^j) \\
&\quad - \frac{i(k_2)^2}{4k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} (\bar{\varepsilon}^i \gamma_* \gamma_m D_\mu \Psi_\nu^j) \mathbf{S}_1^{kl} (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^l) \\
&\quad - \frac{i(k_2)^2 k_4 \sqrt{\Lambda}}{4k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{jh} \mathbf{M}_1^{hi} (\bar{\varepsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^j) \mathbf{S}_1^{kl} (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^l).
\end{aligned}$$

The step (a) justifies the presupposed symmetries $\mathbf{S}_1^T = \mathbf{S}_1$.

The step (b) uses the relations $\gamma_\mu \gamma_* = -\gamma_* \gamma_\mu$ and $\gamma_\mu \gamma^{\alpha\beta} \gamma_* = -\gamma_* \gamma_\mu \gamma^{\alpha\beta}$.

The step (c) uses the results²⁶ $[D_\mu, D_\nu] \Psi_\rho^i = \frac{1}{4} R_{\mu\nu}^{mn} \gamma_{mn} \Psi_\rho^i$ and $[D_\mu, D_\nu] \gamma_\rho = R_{\mu\nu}^{mn} \gamma_m e_{n\rho}$. Therefore one can see that $i \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\bar{\varepsilon}^i \gamma_* \gamma_\nu D_\nu D_\rho \Psi_\sigma^i) + \frac{i}{2} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\bar{\varepsilon}^i \gamma_* D_\mu D_\nu \gamma_\rho \Psi_\sigma^i) =$

²⁶These are calculated in Appendix E.1.

$\frac{i}{8} \int dx^4 \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu}{}^{mn} (\bar{\boldsymbol{\varepsilon}}^i \gamma_* \gamma_\rho \gamma_{mn} \Psi_\sigma^i) + \frac{i}{4} \int dx^4 \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu}{}^{mn} (\bar{\boldsymbol{\varepsilon}}^i \gamma_* \gamma_m \Psi_\sigma^i) e_{n\rho} = \frac{i}{8} \int dx^4 \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu}{}^{mn} e_\rho^r (\bar{\boldsymbol{\varepsilon}}^i \gamma_* \gamma_{mnr} \Psi_\sigma^i)$
with the relation $\gamma_r \gamma_{mn} - e_{mr} \gamma_n + e_{nr} \gamma_m = \gamma_{rmn}$.

The step (c) also uses the relation $\gamma_\mu \gamma_\nu = \gamma_{\mu\nu} + \eta_{\mu\nu}$.

The expression $\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{jh} \mathbf{M}_1^{hi} (\bar{\boldsymbol{\varepsilon}}^i \gamma_* \eta_{\mu\nu} D_\rho \Psi_\sigma^j)$ vanishes in (c) because $\eta_{\mu\nu}$ is symmetric in $\mu\nu$, which clashes with the total antisymmetry of $\epsilon^{\mu\nu\rho\sigma}$.

The step (e) uses the result $D_{[\rho} e_{\sigma]}^m = \frac{k_2}{4k_1} \mathbf{S}_1^{ij} (\bar{\Psi}_\rho^i \gamma^m \Psi_\sigma^j)$.

From (23) we have

$$\begin{aligned}
\delta_\Psi S_{g\Lambda} &= +2k_2 k_4 \sqrt{\Lambda} \delta_\Psi \int dx^4 e \mathbf{S}_2^{ij} (\bar{\Psi}_\mu^i \gamma^{\mu\nu} \Psi_\nu^j), \\
&\stackrel{L}{=} +ik_2 k_4 \sqrt{\Lambda} \delta_\Psi \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\bar{\Psi}_\mu^i \gamma_* \gamma_{\nu\rho} \Psi_\sigma^j), \\
&= +ik_2 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\delta \bar{\Psi}_\mu^i \gamma_* \gamma_{\nu\rho} \Psi_\sigma^j) + ik_2 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\bar{\Psi}_\mu^i \gamma_* \gamma_{\nu\rho} \delta \Psi_\sigma^j), \\
&\stackrel{F}{=} +2ik_2 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\delta \bar{\Psi}_\mu^i \gamma_* \gamma_{\nu\rho} \Psi_\sigma^j), \\
&\stackrel{S}{=} +2ik_2 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} ([D_\mu \bar{\boldsymbol{\varepsilon}}^i - k_4 \sqrt{\Lambda} \mathbf{M}_1^{ih} \bar{\boldsymbol{\varepsilon}}^h \gamma_\mu] \gamma_* \gamma_{\nu\rho} \Psi_\sigma^j), \\
&= +2ik_2 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (D_\mu \bar{\boldsymbol{\varepsilon}}^i \gamma_* \gamma_{\nu\rho} \Psi_\sigma^j) - 2ik_2 (k_4)^2 \Lambda \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{jh} \mathbf{M}_1^{hi} (\bar{\boldsymbol{\varepsilon}}^i \gamma_\mu \gamma_* \gamma_{\nu\rho} \Psi_\sigma^j), \\
&\stackrel{P}{=} -2ik_2 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\bar{\boldsymbol{\varepsilon}}^i \gamma_* D_\mu \gamma_{\nu\rho} \Psi_\sigma^j) - 2ik_2 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\bar{\boldsymbol{\varepsilon}}^i \gamma_* \gamma_{\mu\nu} D_\rho \Psi_\sigma^j) \\
&\quad - 2ik_2 (k_4)^2 \Lambda \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{jh} \mathbf{M}_1^{hi} (\bar{\boldsymbol{\varepsilon}}^i \gamma_\mu \gamma_* \gamma_{\nu\rho} \Psi_\sigma^j), \\
&\stackrel{L}{=} -2ik_2 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\bar{\boldsymbol{\varepsilon}}^i \gamma_* D_\mu \gamma_{\nu\rho} \Psi_\sigma^j) - 4k_2 k_4 \sqrt{\Lambda} \int dx^4 e \mathbf{S}_2^{ij} (\bar{\boldsymbol{\varepsilon}}^i \gamma^{\mu\nu} D_\mu \Psi_\nu^j) \\
&\quad - 4k_2 (k_4)^2 \Lambda \int dx^4 e \mathbf{S}_2^{jh} \mathbf{M}_1^{hi} (\bar{\boldsymbol{\varepsilon}}^i \gamma_\mu \gamma^{\mu\nu} \Psi_\nu^j), \\
&\stackrel{(a)}{=} -2ik_2 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\bar{\boldsymbol{\varepsilon}}^i \gamma_* D_\mu \gamma_{\nu\rho} \Psi_\sigma^j) - 4k_2 k_4 \sqrt{\Lambda} \int dx^4 e \mathbf{S}_2^{ij} (\bar{\boldsymbol{\varepsilon}}^i \gamma^{\mu\nu} D_\mu \Psi_\nu^j) \\
&\quad - 12k_2 (k_4)^2 \Lambda \int dx^4 e \mathbf{S}_2^{jh} \mathbf{M}_1^{hi} (\bar{\boldsymbol{\varepsilon}}^i \gamma^\mu \Psi_\mu^j), \\
&\stackrel{(b)}{=} -2ik_2 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\bar{\boldsymbol{\varepsilon}}^i \gamma_* \gamma_{mn} \Psi_\sigma^j) (D_\mu e_\nu^m e_\rho^n + e_\nu^m D_\mu e_\rho^n) - 4k_2 k_4 \sqrt{\Lambda} \int dx^4 e \mathbf{S}_2^{ij} (\bar{\boldsymbol{\varepsilon}}^i \gamma^{\mu\nu} D_\mu \Psi_\nu^j) \\
&\quad - 12k_2 (k_4)^2 \Lambda \int dx^4 e \mathbf{S}_2^{jh} \mathbf{M}_1^{hi} (\bar{\boldsymbol{\varepsilon}}^i \gamma^\mu \Psi_\mu^j), \\
&= +4ik_2 k_4 \sqrt{\Lambda} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\bar{\boldsymbol{\varepsilon}}^i \gamma_* \gamma_{\mu m} \Psi_\sigma^j) D_\rho e_\sigma^m - 4k_2 k_4 \sqrt{\Lambda} \int dx^4 e \mathbf{S}_2^{ij} (\bar{\boldsymbol{\varepsilon}}^i \gamma^{\mu\nu} D_\mu \Psi_\nu^j) \\
&\quad - 12k_2 (k_4)^2 \Lambda \int dx^4 e \mathbf{S}_2^{jh} \mathbf{M}_1^{hi} (\bar{\boldsymbol{\varepsilon}}^i \gamma^\mu \Psi_\mu^j), \\
&\stackrel{(c)}{=} +\frac{i(k_2)^2 k_4 \sqrt{\Lambda}}{k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} (\bar{\boldsymbol{\varepsilon}}^i \gamma_* \gamma_{\mu m} \Psi_\sigma^j) \mathbf{S}_1^{kl} (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^l) - 4k_2 k_4 \sqrt{\Lambda} \int dx^4 e \mathbf{S}_2^{ij} (\bar{\boldsymbol{\varepsilon}}^i \gamma^{\mu\nu} D_\mu \Psi_\nu^j) \\
&\quad - 12k_2 (k_4)^2 \Lambda \int dx^4 e \mathbf{S}_2^{jh} \mathbf{M}_1^{hi} (\bar{\boldsymbol{\varepsilon}}^i \gamma^\mu \Psi_\mu^j),
\end{aligned}$$

$$\begin{aligned}
&\stackrel{(d)L}{=} -4k_2k_4\sqrt{\Lambda} \int dx^4 e \mathbf{S}_2^{ij}(\bar{\epsilon}^i \gamma^{\mu\nu} D_\mu \Psi_\nu^j) - 12k_2(k_4)^2 \Lambda \int dx^4 e \mathbf{S}_2^{jh} \mathbf{M}_1^{hi}(\bar{\epsilon}^i \gamma^\mu \Psi_\mu^j) \\
&\quad - \frac{i(k_2)^2 k_4 \sqrt{\Lambda}}{k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij}(\bar{\epsilon}^i \gamma_* \Psi_\mu^j) \mathbf{S}_1^{kl}(\bar{\Psi}_\nu^k \gamma_\rho \Psi_\sigma^l) \\
&\quad + \frac{i(k_2)^2 k_4 \sqrt{\Lambda}}{k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij}(\bar{\epsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^j) \mathbf{S}_1^{kl}(\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^l).
\end{aligned}$$

The step (a) uses the relations $\gamma_\mu \gamma^{\mu\nu} = 3\gamma^\nu$ and $\gamma_\mu \gamma^{\alpha\beta} \gamma^{\mu\nu} = -\gamma^{\nu\alpha\beta} - g^{\nu\alpha} \gamma^\beta + g^{\nu\beta} \gamma^\alpha$.

The step (b) uses the definitions $\gamma_{\nu\rho} = \gamma_{mne} e_\nu^m e_\rho^n$.

The step (c) uses the result $D_{[\rho} e_{\sigma]}^m = \frac{k_2}{4k_1} \mathbf{S}_1^{ij}(\bar{\Psi}_\rho^i \gamma^m \Psi_\sigma^j)$.

The step (d) uses the relation $\gamma_{\mu m} = \gamma_\mu \gamma_m - e_{m\mu}$.

D.4 Proof of (38)

$$\begin{aligned}
\delta_e S_{\text{RS}} &\longrightarrow -\frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij}(\bar{\epsilon}^i \gamma_m \Psi_\mu^j) \mathbf{S}_1^{kl}(\bar{\Psi}_\nu^k \gamma_* \gamma^m D_\rho \Psi_\sigma^l), \\
\delta_\Psi S_{\text{RS}} &\longrightarrow -\frac{i(k_2)^2}{4k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij}(\bar{\epsilon}^i \gamma_* \gamma_m D_\mu \Psi_\nu^j) \mathbf{S}_1^{kl}(\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^l). \\
0 &= -\frac{i(k_2)^2}{2k_1} \int \overbrace{dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_1^{kl}(\bar{\epsilon}^i \gamma_m \Psi_\mu^j)(\bar{\Psi}_\nu^k \gamma_* \gamma^m D_\rho \Psi_\sigma^l)}^{} \\
&\quad -\frac{i(k_2)^2}{4k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_1^{kl}(\bar{\epsilon}^i \gamma_* \gamma_m D_\mu \Psi_\nu^j)(\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^l) \\
&\stackrel{R(a)}{=} -\frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_1^{kl}(\bar{\epsilon}^i D_\mu \Psi_\nu^l)(\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^j) + \frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_1^{kl}(\bar{\epsilon}^i \gamma_* D_\mu \Psi_\nu^l)(\bar{\Psi}_\rho^k \Psi_\sigma^j) \\
&\quad + \frac{i(k_2)^2}{4k_1} \int \overbrace{dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_1^{kl}(\bar{\epsilon}^i \gamma_m D_\mu \Psi_\nu^l)(\bar{\Psi}_\rho^k \gamma_* \gamma^m \Psi_\sigma^j)}^{} \\
&\quad + \frac{i(k_2)^2}{4k_1} \int \overbrace{dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_1^{kl}(\bar{\epsilon}^i \gamma_* \gamma_m D_\mu \Psi_\nu^l)(\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^j)}^{} \\
&\quad - \frac{i(k_2)^2}{4k_1} \int \overbrace{dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_1^{kl}(\bar{\epsilon}^i \gamma_* \gamma_m D_\mu \Psi_\nu^l)(\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^l)}, \\
&\stackrel{(b)}{=} -\frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_1^{kl}(\bar{\epsilon}^i D_\mu \Psi_\nu^l)(\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^j) + \frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_1^{kl}(\bar{\epsilon}^i \gamma_* D_\mu \Psi_\nu^l)(\bar{\Psi}_\rho^k \Psi_\sigma^j) \\
&\quad - \frac{i(k_2)^2}{4k_1} \int \overbrace{dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{il} \mathbf{S}_1^{jk})(\bar{\epsilon}^i \gamma_m D_\mu \Psi_\nu^l)(\bar{\Psi}_\rho^k \gamma_* \gamma^m \Psi_\sigma^j)}^{} \\
&\quad + \frac{i(k_2)^2}{4k_1} \int \overbrace{dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{il} \mathbf{S}_1^{jk})(\bar{\epsilon}^i \gamma_* \gamma_m D_\mu \Psi_\nu^l)(\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^j)}, \\
&\stackrel{R(c)}{=} -\frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_1^{kl}(\bar{\epsilon}^i D_\mu \Psi_\nu^l)(\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^j) + \frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_1^{kl}(\bar{\epsilon}^i \gamma_* D_\mu \Psi_\nu^l)(\bar{\Psi}_\rho^k \Psi_\sigma^j) \\
&\quad - \frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{il} \mathbf{S}_1^{jk})(\bar{\epsilon}^i \Psi_\mu^j)(\bar{\Psi}_\nu^k \gamma_* D_\rho \Psi_\sigma^l)
\end{aligned}$$

$$\begin{aligned}
& + \frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{il} \mathbf{S}_1^{jk}) (\bar{\epsilon}^i \gamma_* \Psi_\mu^j) (\bar{\Psi}_\nu^k D_\rho \Psi_\sigma^l), \\
\stackrel{(d)}{=} & - \frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \overbrace{\mathbf{S}_1^{il} \mathbf{S}_1^{jk}}^c (\bar{\epsilon}^i D_\mu \Psi_\nu^j) (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^l) + 2 \int dx^4 \epsilon^{\mu\nu\rho\sigma} \overbrace{\mathbf{S}_1^{il} \mathbf{S}_1^{jk}}^d (\bar{\epsilon}^i \gamma_* D_\mu \Psi_\nu^j) (\bar{\Psi}_\rho^k \Psi_\sigma^l) \\
& - \frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{il} \mathbf{S}_1^{jk}) (\bar{\epsilon}^i \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_* D_\rho \Psi_\sigma^l) \\
& + \frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{il} \mathbf{S}_1^{jk}) (\bar{\epsilon}^i \gamma_* \Psi_\mu^j) (\bar{\Psi}_\nu^k D_\rho \Psi_\sigma^l), \\
= & - \frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{jk} \mathbf{S}_1^{il}) (\bar{\epsilon}^i \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_* D_\rho \Psi_\sigma^l) \\
& + \frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{jk} \mathbf{S}_1^{il}) (\bar{\epsilon}^i \gamma_* \Psi_\mu^j) (\bar{\Psi}_\nu^k D_\rho \Psi_\sigma^l) \\
& - \frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{il} \mathbf{S}_1^{jk} (\bar{\epsilon}^i D_\mu \Psi_\nu^j) (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^l) \\
& + \frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{il} \mathbf{S}_1^{jk} (\bar{\epsilon}^i \gamma_* D_\mu \Psi_\nu^j) (\bar{\Psi}_\rho^k \Psi_\sigma^l), \\
\stackrel{(d)}{=} & - \frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{jk} \mathbf{S}_1^{il}) (\bar{\epsilon}^i \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_* D_\rho \Psi_\sigma^l) \\
& + \frac{i(k_2)^2}{2k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{jk} \mathbf{S}_1^{il}) (\bar{\epsilon}^i \gamma_* D_\mu \Psi_\nu^j) (\bar{\Psi}_\nu^k D_\rho \Psi_\sigma^l) \\
& + \frac{i(k_2)^2}{4k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{jk} \mathbf{S}_1^{il}) (\bar{\epsilon}^i D_\mu \Psi_\nu^j) (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^l) \\
& - \frac{i(k_2)^2}{4k_1} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{jk} \mathbf{S}_1^{il}) (\bar{\epsilon}^i \gamma_* D_\mu \Psi_\nu^j) (\bar{\Psi}_\rho^k \Psi_\sigma^l).
\end{aligned}$$

The step (a) uses the spinor-ansatz reordering $\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_m \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_* \gamma^m D_\rho \Psi_\sigma^l) = \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^j) - \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \Psi_\sigma^j) - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_m D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \gamma^m \Psi_\sigma^j) - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_m D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^j)$.

The step (b) uses the spinor-ansatz flips $\epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\rho^k \gamma_* \gamma^m \Psi_\sigma^j) = -\epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\rho^k \gamma_* \gamma^m \Psi_\sigma^j)$ and $\epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^j) = \epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^j)$.

The step (c) uses the spinor-ansatz reordering $\epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{il} \mathbf{S}_1^{jk}) (\bar{\epsilon}^i \gamma_m D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \gamma^m \Psi_\sigma^j) - \epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{il} \mathbf{S}_1^{jk}) (\bar{\epsilon}^i \gamma_* \gamma_m D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^j) = 2\epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{il} \mathbf{S}_1^{jk}) (\bar{\epsilon}^i \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_* D_\rho \Psi_\sigma^l) - 2\epsilon^{\mu\nu\rho\sigma} (\mathbf{S}_1^{ik} \mathbf{S}_1^{jl} - \mathbf{S}_1^{il} \mathbf{S}_1^{jk}) (\bar{\epsilon}^i \gamma_* \Psi_\mu^j) (\bar{\Psi}_\nu^k D_\rho \Psi_\sigma^l)$.

The step (d) uses the spinor-ansatz flips $\epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^l) = -\epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\rho^l \gamma_* \Psi_\sigma^k)$ and $\epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\rho^k \Psi_\sigma^l) = -\epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\rho^l \Psi_\sigma^k)$.

Fierz 1

$$\begin{aligned}
(\bar{\chi}_1 \chi_2) (\bar{\chi}_3 \chi_4) &= -\frac{1}{4} (\bar{\chi}_3 \chi_2) (\bar{\chi}_1 \chi_4) - \frac{1}{4} (\bar{\chi}_3 \gamma_* \chi_2) (\bar{\chi}_1 \gamma_* \chi_4) \\
&\quad - \frac{1}{4} (\bar{\chi}_3 \gamma_a \chi_2) (\bar{\chi}_1 \gamma^a \chi_4) + \frac{1}{4} (\bar{\chi}_3 \gamma_* \gamma_a \chi_2) (\bar{\chi}_1 \gamma_* \gamma^a \chi_4)
\end{aligned}$$

$$+\frac{1}{8}(\bar{\chi}_3\gamma_{ab}\chi_2)(\bar{\chi}_1\gamma^{ab}\chi_4)$$

$$\bar{\chi}_1 = \bar{\Psi}_\nu^k \gamma_* \gamma^m, \chi_2 = D_\rho \Psi_\sigma^l, \bar{\chi}_3 = \bar{\epsilon}^i, \chi_4 = \gamma_m \Psi_\mu^j$$

$$\begin{aligned} \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_1^{kl} (\bar{\Psi}_\nu^k \gamma_* \gamma^m D_\rho \Psi_\sigma^l) (\bar{\epsilon}^i \gamma_m \Psi_\mu^j) &= -\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_1^{kl} (\bar{\epsilon}^i D_\rho \Psi_\sigma^l) (\bar{\Psi}_\nu^k \gamma_* \gamma^m \gamma_m \Psi_\mu^j) \\ &\quad -\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_1^{kl} (\bar{\epsilon}^i \gamma_* D_\rho \Psi_\sigma^l) (\bar{\Psi}_\nu^k \gamma_* \gamma^m \gamma_* \gamma_m \Psi_\mu^j) \\ &\quad -\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_1^{kl} (\bar{\epsilon}^i \gamma_a D_\rho \Psi_\sigma^l) (\bar{\Psi}_\nu^k \gamma_* \gamma^m \gamma_* \gamma^a \gamma_m \Psi_\mu^j) \\ &\quad +\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_a D_\rho \Psi_\sigma^l) (\bar{\Psi}_\nu^k \gamma_* \gamma^m \gamma_* \gamma^a \gamma_m \Psi_\mu^j) \\ &\quad +\frac{1}{8} \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_1^{kl} (\bar{\epsilon}^i \gamma_{ab} D_\rho \Psi_\sigma^l) (\bar{\Psi}_\nu^k \gamma_* \gamma^m \gamma_* \gamma^{ab} \gamma_m \Psi_\mu^j), \\ &= +\epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_1^{kl} (\bar{\epsilon}^i D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^j) - \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_1^{kl} (\bar{\epsilon}^i \gamma_* D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \Psi_\sigma^j) \\ &\quad -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_1^{kl} (\bar{\epsilon}^i \gamma_m D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \gamma^m \Psi_\sigma^j) \\ &\quad -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_m D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^j), \end{aligned}$$

$$(1) \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_1^{kl} (\bar{\epsilon}^i \gamma_m \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_* \gamma^m D_\rho \Psi_\sigma^l) = \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_1^{kl} (\bar{\epsilon}^i D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^j) - \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_1^{kl} (\bar{\epsilon}^i \gamma_* D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \Psi_\sigma^j) \\ - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_1^{kl} (\bar{\epsilon}^i \gamma_m D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \gamma^m \Psi_\sigma^j) - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_m D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^j).$$

Fierz 2a

$$\begin{aligned} (\bar{\chi}_1 \chi_2) (\bar{\chi}_3 \chi_4) &= -\frac{1}{4} (\bar{\chi}_3 \chi_2) (\bar{\chi}_1 \chi_4) - \frac{1}{4} (\bar{\chi}_3 \gamma_* \chi_2) (\bar{\chi}_1 \gamma_* \chi_4) \\ &\quad -\frac{1}{4} (\bar{\chi}_3 \gamma_a \chi_2) (\bar{\chi}_1 \gamma^a \chi_4) + \frac{1}{4} (\bar{\chi}_3 \gamma_* \gamma_a \chi_2) (\bar{\chi}_1 \gamma_* \gamma^a \chi_4) \\ &\quad +\frac{1}{8} (\bar{\chi}_3 \gamma_{ab} \chi_2) (\bar{\chi}_1 \gamma^{ab} \chi_4) \end{aligned}$$

$$\bar{\chi}_1 = \bar{\Psi}_\nu^k \gamma_*, \chi_2 = D_\rho \Psi_\sigma^l, \bar{\chi}_3 = \bar{\epsilon}^i, \chi_4 = \Psi_\mu^j$$

$$\begin{aligned} \epsilon^{\mu\nu\rho\sigma} (S_1^{ik} S_1^{jl} - S_1^{il} S_1^{jk}) (\bar{\Psi}_\nu^k \gamma_* D_\rho \Psi_\sigma^l) (\bar{\epsilon}^i \Psi_\mu^j) &= -\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} (S_1^{ik} S_1^{jl} - S_1^{il} S_1^{jk}) (\bar{\epsilon}^i D_\rho \Psi_\sigma^l) (\bar{\Psi}_\nu^k \gamma_* \Psi_\mu^j) \\ &\quad -\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} (S_1^{ik} S_1^{jl} - S_1^{il} S_1^{jk}) (\bar{\epsilon}^i \gamma_* D_\rho \Psi_\sigma^l) (\bar{\Psi}_\nu^k \gamma_* \gamma_* \Psi_\mu^j) \\ &\quad -\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} (S_1^{ik} S_1^{jl} - S_1^{il} S_1^{jk}) (\bar{\epsilon}^i \gamma_a D_\rho \Psi_\sigma^l) (\bar{\Psi}_\nu^k \gamma_* \gamma^a \Psi_\mu^j) \\ &\quad +\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} (S_1^{ik} S_1^{jl} - S_1^{il} S_1^{jk}) (\bar{\epsilon}^i \gamma_* \gamma_a D_\rho \Psi_\sigma^l) (\bar{\Psi}_\nu^k \gamma_* \gamma_* \gamma^a \Psi_\mu^j) \\ &\quad +\frac{1}{8} \epsilon^{\mu\nu\rho\sigma} (S_1^{ik} S_1^{jl} - S_1^{il} S_1^{jk}) (\bar{\epsilon}^i \gamma_{ab} D_\rho \Psi_\sigma^l) (\bar{\Psi}_\nu^k \gamma_* \gamma^{ab} \Psi_\mu^j) \\ &= +\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} (S_1^{ik} S_1^{jl} - S_1^{il} S_1^{jk}) (\bar{\epsilon}^i D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^j) \\ &\quad +\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} (S_1^{ik} S_1^{jl} - S_1^{il} S_1^{jk}) (\bar{\epsilon}^i \gamma_* D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \Psi_\sigma^j) \\ &\quad +\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} (S_1^{ik} S_1^{jl} - S_1^{il} S_1^{jk}) (\bar{\epsilon}^i \gamma_m D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \gamma^m \Psi_\sigma^j) \end{aligned}$$

$$-\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(S_1^{ik}S_1^{jl}-S_1^{il}S_1^{jk})(\bar{\varepsilon}^i\gamma_*\gamma_mD_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^k\gamma^m\Psi_\sigma^j) \\ -\frac{1}{8}\epsilon^{\mu\nu\rho\sigma}(S_1^{ik}S_1^{jl}-S_1^{il}S_1^{jk})(\bar{\varepsilon}^i\gamma_{mn}D_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^k\gamma_*\gamma^{mn}\Psi_\sigma^j),$$

$$(2a) \quad \epsilon^{\mu\nu\rho\sigma} (S_1^{ik} S_1^{jl} - S_1^{il} S_1^{jk}) (\bar{\epsilon}^i \Psi_\mu^j) (\bar{\Psi}_\rho^k \gamma_* D_\mu \Psi_\sigma^l) = + \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} (S_1^{ik} S_1^{jl} - S_1^{il} S_1^{jk}) (\bar{\epsilon}^i D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^j) \\ + \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} (S_1^{ik} S_1^{jl} - S_1^{il} S_1^{jk}) (\bar{\epsilon}^i \gamma_* D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \Psi_\sigma^j) + \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} (S_1^{ik} S_1^{jl} - S_1^{il} S_1^{jk}) (\bar{\epsilon}^i \gamma_m D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \gamma^m \Psi_\sigma^j) \\ - \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} (S_1^{ik} S_1^{jl} - S_1^{il} S_1^{jk}) (\bar{\epsilon}^i \gamma_* \gamma_m D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^j) - \frac{1}{8} \epsilon^{\mu\nu\rho\sigma} (S_1^{ik} S_1^{jl} - S_1^{il} S_1^{jk}) (\bar{\epsilon}^i \gamma_{mn} D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \gamma^{mn} \Psi_\sigma^j)$$

Fierz 2b

$$\begin{aligned}
(\bar{\chi}_1 \chi_2)(\bar{\chi}_3 \chi_4) &= -\frac{1}{4} (\bar{\chi}_3 \chi_2)(\bar{\chi}_1 \chi_4) - \frac{1}{4} (\bar{\chi}_3 \gamma_* \chi_2)(\bar{\chi}_1 \gamma_* \chi_4) \\
&\quad - \frac{1}{4} (\bar{\chi}_3 \gamma_a \chi_2)(\bar{\chi}_1 \gamma^a \chi_4) + \frac{1}{4} (\bar{\chi}_3 \gamma_* \gamma_a \chi_2)(\bar{\chi}_1 \gamma_* \gamma^a \chi_4) \\
&\quad + \frac{1}{8} (\bar{\chi}_3 \gamma_{ab} \chi_2)(\bar{\chi}_1 \gamma^{ab} \chi_4)
\end{aligned}$$

$$\bar{\chi}_1 = \bar{\Psi}_{\nu}^k, \chi_2 = D_{\rho} \Psi_{\sigma}^l, \bar{\chi}_3 = \bar{\varepsilon}^i, \chi_4 = \gamma_* \Psi_{\mu}^j$$

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma}(\mathbf{S}_1^{ik}\mathbf{S}_1^{jl} - \mathbf{S}_1^{il}\mathbf{S}_1^{jk})(\bar{\Psi}_\nu D_\rho \Psi_\sigma^l)(\bar{\varepsilon}^i \gamma_* \Psi_\mu^j) &= -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\mathbf{S}_1^{ik}\mathbf{S}_1^{jl} - \mathbf{S}_1^{il}\mathbf{S}_1^{jk})(\bar{\varepsilon}^i D_\rho \Psi_\sigma^l)(\bar{\Psi}_\nu^k \gamma_* \Psi_\mu^j) \\
&\quad -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\mathbf{S}_1^{ik}\mathbf{S}_1^{jl} - \mathbf{S}_1^{il}\mathbf{S}_1^{jk})(\bar{\varepsilon}^i \gamma_* D_\rho \Psi_\sigma^l)(\bar{\Psi}_\nu^k \gamma_* \gamma_* \Psi_\mu^j) \\
&\quad -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\mathbf{S}_1^{ik}\mathbf{S}_1^{jl} - \mathbf{S}_1^{il}\mathbf{S}_1^{jk})(\bar{\varepsilon}^i \gamma_a D_\rho \Psi_\sigma^l)(\bar{\Psi}_\nu^k \gamma^a \gamma_* \Psi_\mu^j) \\
&\quad +\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\mathbf{S}_1^{ik}\mathbf{S}_1^{jl} - \mathbf{S}_1^{il}\mathbf{S}_1^{jk})(\bar{\varepsilon}^i \gamma_* \gamma_a D_\rho \Psi_\sigma^l)(\bar{\Psi}_\nu^k \gamma_* \gamma^a \gamma_* \Psi_\mu^j) \\
&\quad +\frac{1}{8}\epsilon^{\mu\nu\rho\sigma}(\mathbf{S}_1^{ik}\mathbf{S}_1^{jl} - \mathbf{S}_1^{il}\mathbf{S}_1^{jk})(\bar{\varepsilon}^i \gamma_{ab} D_\rho \Psi_\sigma^l)(\bar{\Psi}_\nu^k \gamma^{ab} \gamma_* \Psi_\mu^j), \\
&= +\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\mathbf{S}_1^{ik}\mathbf{S}_1^{jl} - \mathbf{S}_1^{il}\mathbf{S}_1^{jk})(\bar{\varepsilon}^i D_\mu \Psi_\nu^l)(\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^j) \\
&\quad +\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\mathbf{S}_1^{ik}\mathbf{S}_1^{jl} - \mathbf{S}_1^{il}\mathbf{S}_1^{jk})(\bar{\varepsilon}^i \gamma_* D_\mu \Psi_\nu^l)(\bar{\Psi}_\rho^k \Psi_\sigma^j) \\
&\quad -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\mathbf{S}_1^{ik}\mathbf{S}_1^{jl} - \mathbf{S}_1^{il}\mathbf{S}_1^{jk})(\bar{\varepsilon}^i \gamma_m D_\mu \Psi_\nu^l)(\bar{\Psi}_\rho^k \gamma_* \gamma^m \Psi_\sigma^j) \\
&\quad +\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\mathbf{S}_1^{ik}\mathbf{S}_1^{jl} - \mathbf{S}_1^{il}\mathbf{S}_1^{jk})(\bar{\varepsilon}^i \gamma_* \gamma_m D_\mu \Psi_\nu^l)(\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^j) \\
&\quad -\frac{1}{8}\epsilon^{\mu\nu\rho\sigma}(\mathbf{S}_1^{ik}\mathbf{S}_1^{jl} - \mathbf{S}_1^{il}\mathbf{S}_1^{jk})(\bar{\varepsilon}^i \gamma_{mn} D_\mu \Psi_\nu^l)(\bar{\Psi}_\rho^k \gamma_* \gamma^{mn} \Psi_\sigma^j),
\end{aligned}$$

$$(2b) \quad \epsilon^{\mu\nu\rho\sigma}(S_1^{ik}S_1^{jl}-S_1^{il}S_1^{jk})(\bar{\epsilon}^i\gamma_*\Psi_\mu^j)(\bar{\Psi}_\nu^kD_\rho\Psi_\sigma^l)=+\tfrac{1}{4}\epsilon^{\mu\nu\rho\sigma}(S_1^{ik}S_1^{jl}-S_1^{il}S_1^{jk})(\bar{\epsilon}^iD_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^k\gamma_*\Psi_\sigma^j) \\ +\tfrac{1}{4}\epsilon^{\mu\nu\rho\sigma}(S_1^{ik}S_1^{jl}-S_1^{il}S_1^{jk})(\bar{\epsilon}^i\gamma_*D_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^k\Psi_\sigma^j)-\tfrac{1}{4}\epsilon^{\mu\nu\rho\sigma}(S_1^{ik}S_1^{jl}-S_1^{il}S_1^{jk})(\bar{\epsilon}^i\gamma_mD_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^k\gamma_*\gamma^m\Psi_\sigma^j) \\ +\tfrac{1}{4}\epsilon^{\mu\nu\rho\sigma}(S_1^{ik}S_1^{jl}-S_1^{il}S_1^{jk})(\bar{\epsilon}^i\gamma_*\gamma_mD_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^k\gamma^m\Psi_\sigma^j)-\tfrac{1}{8}\epsilon^{\mu\nu\rho\sigma}(S_1^{ik}S_1^{jl}-S_1^{il}S_1^{jk})(\bar{\epsilon}^i\gamma_{mn}D_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^k\gamma_*\gamma^{mn}\Psi_\sigma^j).$$

$$(2a) - (2b) \epsilon^{\mu\nu\rho\sigma} (S_1^{ik} S_1^{jl} - S_1^{il} S_1^{jk}) (\bar{\epsilon}^i \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_* D_\rho \Psi_\sigma^l) - \epsilon^{\mu\nu\rho\sigma} (S_1^{ik} S_1^{jl} - S_1^{il} S_1^{jk}) (\bar{\epsilon}^i \gamma_* \Psi_\mu^j) (\bar{\Psi}_\nu^k D_\rho \Psi_\sigma^l) = \\ + \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} (S_1^{ik} S_1^{jl} - S_1^{il} S_1^{jk}) (\bar{\epsilon}^i \gamma_m D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \gamma^m \Psi_\sigma^j) - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} (S_1^{ik} S_1^{jl} - S_1^{il} S_1^{jk}) (\bar{\epsilon}^i \gamma_* \gamma_m D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^j)$$

$$\Rightarrow (2) \epsilon^{\mu\nu\rho\sigma} (S_1^{ik} S_1^{jl} - S_1^{il} S_1^{jk}) (\bar{\varepsilon}^i \gamma_m D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \gamma^m \Psi_\sigma^j) - \epsilon^{\mu\nu\rho\sigma} (S_1^{ik} S_1^{jl} - S_1^{il} S_1^{jk}) (\bar{\varepsilon}^i \gamma_* \gamma_m D_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^j) = \\ 2 \epsilon^{\mu\nu\rho\sigma} (S_1^{ik} S_1^{jl} - S_1^{il} S_1^{jk}) (\bar{\varepsilon}^i \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_* D_\rho \Psi_\sigma^l) - 2 \epsilon^{\mu\nu\rho\sigma} (S_1^{ik} S_1^{jl} - S_1^{il} S_1^{jk}) (\bar{\varepsilon}^i \gamma_* \Psi_\mu^j) (\bar{\Psi}_\nu^k D_\rho \Psi_\sigma^l).$$

D.5 Proof of (39)

$$\begin{aligned}
& -\frac{i(k_2)^2\sqrt{\Lambda}}{4k_1\sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (2S_1^{il}S_2^{jk} - S_1^{jk}S_2^{il})(\bar{\epsilon}^i\gamma_\mu\Psi_\nu^j)(\bar{\Psi}_\rho^k\gamma_*\Psi_\sigma^l) \\
& + \frac{i(k_2)^2\sqrt{\Lambda}}{4k_1\sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (2S_1^{il}S_2^{jk} + S_1^{jk}S_2^{il})(\bar{\epsilon}^i\gamma_*\gamma_\mu\Psi_\nu^j)(\bar{\Psi}_\rho^k\Psi_\sigma^l), \\
& \stackrel{(f)}{=} -\frac{i(k_2)^2\sqrt{\Lambda}}{4k_1\sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (S_1^{ik}S_2^{jl} - S_1^{jk}S_2^{il} - S_1^{il}S_2^{jk} + S_1^{jl}S_2^{ik})(\bar{\epsilon}^i\gamma_\mu\Psi_\nu^j)(\bar{\Psi}_\rho^k\gamma_*\gamma_\rho\Psi_\sigma^l) \\
& + \frac{i(k_2)^2\sqrt{\Lambda}}{4k_1\sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (S_1^{ik}S_2^{jl} - S_1^{jk}S_2^{il} + S_1^{il}S_2^{jk} - S_1^{jl}S_2^{ik})(\bar{\epsilon}^i\gamma_*\Psi_\mu^j)(\bar{\Psi}_\nu^k\gamma_\rho\Psi_\sigma^l) \\
& + \frac{i(k_2)^2\sqrt{\Lambda}}{8k_1\sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (2S_1^{ik}S_2^{jl} - 2S_1^{il}S_2^{jk} - S_1^{jl}S_2^{ik} + S_1^{jk}S_2^{il})(\bar{\epsilon}^i\gamma_\mu\Psi_\nu^j)(\bar{\Psi}_\rho^k\gamma_*\Psi_\sigma^l) \\
& - \frac{i(k_2)^2\sqrt{\Lambda}}{8k_1\sqrt{3}} \int dx^4 \epsilon^{\mu\nu\rho\sigma} (2S_1^{ik}S_2^{jl} - 2S_1^{il}S_2^{jk} + S_1^{jl}S_2^{ik} - S_1^{jk}S_2^{il})(\bar{\epsilon}^i\gamma_*\gamma_\mu\Psi_\nu^j)(\bar{\Psi}_\rho^k\Psi_\sigma^l).
\end{aligned}$$

The step (a) uses the spinor-ansatz reordering $\epsilon^{\mu\nu\rho\sigma} S_1^{ij}S_2^{kl}(\bar{\epsilon}^i\gamma_m\Psi_\mu^j)(\bar{\Psi}_\nu^k\gamma_*\gamma^m\gamma_\rho\Psi_\sigma^l) = -\epsilon^{\mu\nu\rho\sigma} S_1^{ij}S_2^{kl}(\bar{\epsilon}^i\gamma_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^j\gamma_*\Psi_\sigma^k) + \epsilon^{\mu\nu\rho\sigma} S_1^{ij}S_2^{kl}(\bar{\epsilon}^i\gamma_*\gamma_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^j\Psi_\sigma^k) + \epsilon^{\mu\nu\rho\sigma} S_1^{ij}S_2^{kl}(\bar{\epsilon}^i\Psi_\mu^k)(\bar{\Psi}_\nu^j\gamma_*\gamma_\rho\Psi_\sigma^l) - \epsilon^{\mu\nu\rho\sigma} S_1^{ij}S_2^{kl}(\bar{\epsilon}^i\gamma_*\Psi_\mu^k)(\bar{\Psi}_\nu^j\gamma_\rho\Psi_\sigma^l)$.

The step (b) uses the spinor-ansatz reordering $\epsilon^{\mu\nu\rho\sigma} S_2^{ij}S_1^{kl}(\bar{\epsilon}^i\gamma_*\gamma_\mu\gamma_m\Psi_\nu^j)(\bar{\Psi}_\rho^k\gamma^m\Psi_\sigma^l) = -2\epsilon^{\mu\nu\rho\sigma} S_2^{ij}S_1^{kl}(\bar{\epsilon}^i\gamma_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^j\gamma_*\Psi_\sigma^k) - 2\epsilon^{\mu\nu\rho\sigma} S_2^{ij}S_1^{kl}(\bar{\epsilon}^i\gamma_*\gamma_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^j\Psi_\sigma^k)$.

The step (c) uses the spinor-ansatz reordering $\epsilon^{\mu\nu\rho\sigma} S_2^{ij}S_1^{kl}(\bar{\epsilon}^i\gamma_*\Psi_\mu^j)(\bar{\Psi}_\nu^k\gamma_\rho\Psi_\sigma^l) = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} S_2^{ij}S_1^{kl}(\bar{\epsilon}^i\gamma_m\Psi_\mu^l)(\bar{\Psi}_\nu^j\gamma_*\gamma^m\gamma_\rho\Psi_\sigma^k) - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} S_2^{ij}S_1^{kl}(\bar{\epsilon}^i\gamma_*\gamma_m\Psi_\mu^l)(\bar{\Psi}_\nu^j\gamma^m\gamma_\rho\Psi_\sigma^k)$.

The step (d) uses the spinor-ansatz reordering $\epsilon^{\mu\nu\rho\sigma} S_2^{ij}S_1^{kl}(\bar{\epsilon}^i\gamma_m\Psi_\mu^l)(\bar{\Psi}_\nu^j\gamma_*\gamma^m\gamma_\rho\Psi_\sigma^k) + \epsilon^{\mu\nu\rho\sigma} S_2^{ij}S_1^{kl}(\bar{\epsilon}^i\gamma_*\gamma_m\Psi_\mu^l)(\bar{\Psi}_\nu^j\gamma^m\gamma_\rho\Psi_\sigma^k) = -2\epsilon^{\mu\nu\rho\sigma} S_2^{ij}S_1^{kl}(\bar{\epsilon}^i\Psi_\mu^k)(\bar{\Psi}_\nu^j\gamma_*\gamma_\rho\Psi_\sigma^l) - 2\epsilon^{\mu\nu\rho\sigma} S_2^{ij}S_1^{kl}(\bar{\epsilon}^i\gamma_*\Psi_\mu^k)(\bar{\Psi}_\nu^j\gamma_\rho\Psi_\sigma^l) + \epsilon^{\mu\nu\rho\sigma} S_2^{ij}S_1^{kl}(\bar{\epsilon}^i\gamma_\mu\gamma_m\Psi_\nu^l)(\bar{\Psi}_\rho^j\gamma_*\gamma^m\Psi_\sigma^l) + \epsilon^{\mu\nu\rho\sigma} S_2^{ij}S_1^{kl}(\bar{\epsilon}^i\gamma_*\gamma_\mu\gamma_m\Psi_\nu^l)(\bar{\Psi}_\rho^j\gamma^m\Psi_\sigma^l)$.

The step (e) uses the spinor-ansatz reordering $\epsilon^{\mu\nu\rho\sigma} S_2^{ij}S_1^{kl}(\bar{\epsilon}^i\gamma_\mu\gamma_m\Psi_\nu^l)(\bar{\Psi}_\rho^j\gamma_*\gamma^m\Psi_\sigma^l) + \epsilon^{\mu\nu\rho\sigma} S_2^{ij}S_1^{kl}(\bar{\epsilon}^i\gamma_*\gamma_\mu\gamma_m\Psi_\nu^l)(\bar{\Psi}_\rho^j\gamma^m\Psi_\sigma^l) = +2\epsilon^{\mu\nu\rho\sigma} S_2^{ij}S_1^{kl}(\bar{\epsilon}^i\gamma_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^j\gamma_*\Psi_\sigma^k) + 2\epsilon^{\mu\nu\rho\sigma} S_2^{ij}S_1^{kl}(\bar{\epsilon}^i\gamma_*\gamma_\mu\Psi_\nu^l)(\bar{\Psi}_\rho^j\Psi_\sigma^k)$.

The step (f) uses the spinor-ansatz flips $\epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\nu^k\gamma_*\gamma_\rho\Psi_\sigma^l) = -\epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\nu^l\gamma_*\gamma_\rho\Psi_\sigma^k)$, $\epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\nu^k\gamma_\rho\Psi_\sigma^l) = \epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\nu^l\gamma_\rho\Psi_\sigma^k)$, $\epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\rho^k\gamma_*\Psi_\sigma^l) = -\epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\rho^l\gamma_*\Psi_\sigma^k)$ and $\epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\rho^k\Psi_\sigma^l) = -\epsilon^{\mu\nu\rho\sigma} (\bar{\Psi}_\rho^l\Psi_\sigma^k)$.

Fierz 1a

$$\begin{aligned}
(\bar{\chi}_1\chi_2)(\bar{\chi}_3\chi_4) &= -\frac{1}{4}(\bar{\chi}_3\chi_2)(\bar{\chi}_1\chi_4) - \frac{1}{4}(\bar{\chi}_3\gamma_*\chi_2)(\bar{\chi}_1\gamma_*\chi_4) \\
&\quad -\frac{1}{4}(\bar{\chi}_3\gamma_a\chi_2)(\bar{\chi}_1\gamma^a\chi_4) + \frac{1}{4}(\bar{\chi}_3\gamma_*\gamma_a\chi_2)(\bar{\chi}_1\gamma_*\gamma^a\chi_4) \\
&\quad +\frac{1}{8}(\bar{\chi}_3\gamma_{ab}\chi_2)(\bar{\chi}_1\gamma^{ab}\chi_4)
\end{aligned}$$

$$\bar{\chi}_1 = \bar{\Psi}_\nu^k\gamma_*\gamma^m, \chi_2 = \gamma_\rho\Psi_\sigma^l, \bar{\chi}_3 = \bar{\epsilon}^i, \chi_4 = \gamma_m\Psi_\mu^j$$

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma} S_1^{ij}S_2^{kl}(\bar{\Psi}_\nu^k\gamma_*\gamma^m\gamma_\rho\Psi_\sigma^l)(\bar{\epsilon}^i\gamma_m\Psi_\mu^j) &= -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma} S_1^{ij}S_2^{kl}(\bar{\epsilon}^i\gamma_\rho\Psi_\sigma^l)(\bar{\Psi}_\nu^k\gamma_*\gamma^m\gamma_m\Psi_\mu^j) \\
&\quad -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma} S_1^{ij}S_2^{kl}(\bar{\epsilon}^i\gamma_*\gamma_\rho\Psi_\sigma^l)(\bar{\Psi}_\nu^k\gamma_*\gamma^m\gamma_*\gamma_m\Psi_\mu^j) \\
&\quad -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma} S_1^{ij}S_2^{kl}(\bar{\epsilon}^i\gamma_a\gamma_\rho\Psi_\sigma^l)(\bar{\Psi}_\nu^k\gamma_*\gamma^m\gamma^a\gamma_m\Psi_\mu^j)
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_2^{kl} (\bar{\epsilon}^i \gamma_* \gamma_a \gamma_\rho \Psi_\sigma^l) (\bar{\Psi}_\nu^k \gamma_* \gamma^m \gamma_* \gamma^a \gamma_m \Psi_\mu^j) \\
& + \frac{1}{8} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_2^{kl} (\bar{\epsilon}^i \gamma_{ab} \gamma_\rho \Psi_\sigma^l) (\bar{\Psi}_\nu^k \gamma_* \gamma^m \gamma_* \gamma^{ab} \gamma_m \overline{\Psi_\mu^j}), \\
= & + \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_2^{kl} (\bar{\epsilon}^i \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^j) \\
& - \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_2^{kl} (\bar{\epsilon}^i \gamma_* \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \Psi_\sigma^j) \\
& - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_2^{kl} (\bar{\epsilon}^i \gamma_m \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \gamma^m \Psi_\sigma^j) \\
& - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_2^{kl} (\bar{\epsilon}^i \gamma_* \gamma_m \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^j), \\
\stackrel{F}{=} & + \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_2^{kl} (\bar{\epsilon}^i \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^j) \\
& - \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_2^{kl} (\bar{\epsilon}^i \gamma_* \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \Psi_\sigma^j) \\
& + \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_2^{kl} (\bar{\epsilon}^i \gamma_m \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \Psi_\sigma^k) \\
& - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_2^{kl} (\bar{\epsilon}^i \gamma_* \gamma_m \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma^m \Psi_\sigma^k),
\end{aligned}$$

$$\begin{aligned}
(1a) \quad & \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_2^{kl} (\bar{\epsilon}^i \gamma_m \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_* \gamma^m \gamma_\mu \Psi_\sigma^l) = \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_2^{kl} (\bar{\epsilon}^i \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^j) - \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_2^{kl} (\bar{\epsilon}^i \gamma_* \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \Psi_\sigma^j) \\
& + \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_2^{kl} (\bar{\epsilon}^i \gamma_m \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \Psi_\sigma^k) - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_2^{kl} (\bar{\epsilon}^i \gamma_* \gamma_m \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma^m \Psi_\sigma^k).
\end{aligned}$$

Fierz 1b

$$\begin{aligned}
(\bar{\chi}_1 \chi_2)(\bar{\chi}_3 \chi_4) = & -\frac{1}{4} (\bar{\chi}_3 \chi_2)(\bar{\chi}_1 \chi_4) - \frac{1}{4} (\bar{\chi}_3 \gamma_* \chi_2)(\bar{\chi}_1 \gamma_* \chi_4) \\
& - \frac{1}{4} (\bar{\chi}_3 \gamma_a \chi_2)(\bar{\chi}_1 \gamma^a \chi_4) + \frac{1}{4} (\bar{\chi}_3 \gamma_* \gamma_a \chi_2)(\bar{\chi}_1 \gamma_* \gamma^a \chi_4) \\
& + \frac{1}{8} (\bar{\chi}_3 \gamma_{ab} \chi_2)(\bar{\chi}_1 \gamma^{ab} \chi_4)
\end{aligned}$$

$$\bar{\chi}_1 = \bar{\Psi}_\rho^j \gamma_* \gamma^m, \chi_2 = \Psi_\sigma^k, \bar{\chi}_3 = \bar{\epsilon}^i, \chi_4 = \gamma_m \gamma_\mu \Psi_\nu^l$$

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_2^{kl} (\bar{\Psi}_\rho^j \gamma_* \gamma^m \Psi_\sigma^k) (\bar{\epsilon}^i \gamma_m \gamma_\mu \Psi_\nu^l) = & -\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_2^{kl} (\bar{\epsilon}^i \Psi_\sigma^k) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \gamma_m \gamma_\mu \Psi_\nu^l) \\
& - \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_2^{kl} (\bar{\epsilon}^i \gamma_* \Psi_\sigma^k) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \gamma_* \gamma_m \gamma_\mu \Psi_\nu^l) \\
& - \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_2^{kl} (\bar{\epsilon}^i \gamma_a \Psi_\sigma^k) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \gamma^a \gamma_m \gamma_\mu \Psi_\nu^l) \\
& + \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_2^{kl} (\bar{\epsilon}^i \gamma_* \gamma_a \Psi_\sigma^k) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \gamma_* \gamma^a \gamma_m \gamma_\mu \Psi_\nu^l) \\
& + \frac{1}{8} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_2^{kl} (\bar{\epsilon}^i \gamma_{ab} \Psi_\sigma^k) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \gamma_* \gamma^{ab} \gamma_m \gamma_\mu \overline{\Psi_\nu^l}), \\
= & + \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_2^{kl} (\bar{\epsilon}^i \Psi_\mu^k) (\bar{\Psi}_\nu^j \gamma_* \gamma_\rho \Psi_\sigma^l) \\
& - \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_2^{kl} (\bar{\epsilon}^i \gamma_* \Psi_\mu^k) (\bar{\Psi}_\nu^j \gamma_\rho \Psi_\sigma^l) \\
& - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_2^{kl} (\bar{\epsilon}^i \gamma_m \Psi_\mu^k) (\bar{\Psi}_\nu^j \gamma_* \gamma^m \gamma_\rho \Psi_\sigma^l) \\
& - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_1^{ij} \mathbf{S}_2^{kl} (\bar{\epsilon}^i \gamma_* \gamma_m \Psi_\mu^k) (\bar{\Psi}_\nu^j \gamma^m \gamma_\rho \Psi_\sigma^l),
\end{aligned}$$

$$(1b) \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_2^{kl} (\bar{\varepsilon}^i \gamma_m \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \Psi_\sigma^k) = \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_2^{kl} (\bar{\varepsilon}^i \Psi_\mu^k) (\bar{\Psi}_\nu^j \gamma_* \gamma_\rho \Psi_\sigma^l) - \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_2^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_m \Psi_\mu^k) (\bar{\Psi}_\nu^j \gamma^m \gamma_\rho \Psi_\sigma^l) - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_2^{kl} (\bar{\varepsilon}^i \gamma_m \Psi_\mu^k) (\bar{\Psi}_\nu^j \gamma_* \gamma^m \gamma_\rho \Psi_\sigma^l) - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_2^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_m \Psi_\mu^k) (\bar{\Psi}_\nu^j \gamma^m \gamma_\rho \Psi_\sigma^l)$$

Fierz 1c

$$\begin{aligned}
(\bar{\chi}_1 \chi_2)(\bar{\chi}_3 \chi_4) = & -\frac{1}{4}(\bar{\chi}_3 \chi_2)(\bar{\chi}_1 \chi_4) - \frac{1}{4}(\bar{\chi}_3 \gamma_* \chi_2)(\bar{\chi}_1 \gamma_* \chi_4) \\
& -\frac{1}{4}(\bar{\chi}_3 \gamma_a \chi_2)(\bar{\chi}_1 \gamma^a \chi_4) + \frac{1}{4}(\bar{\chi}_3 \gamma_* \gamma_a \chi_2)(\bar{\chi}_1 \gamma_* \gamma^a \chi_4) \\
& +\frac{1}{8}(\bar{\chi}_3 \gamma_{ab} \chi_2)(\bar{\chi}_1 \gamma^{ab} \chi_4)
\end{aligned}$$

$$\bar{\chi}_1 = \bar{\Psi}_{\rho}^j \gamma^m, \chi_2 = \Psi_{\sigma}^k, \bar{\chi}_3 = \bar{\varepsilon}^i, \chi_4 = \gamma_* \gamma_m \gamma_{\mu} \Psi_{\nu}^l$$

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_2^{kl} (\bar{\Psi}_\rho^j \gamma^m \Psi_\sigma^k) (\bar{\varepsilon}^i \gamma_* \gamma_m \gamma_\mu \Psi_\nu^l) &= -\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_2^{kl} (\bar{\varepsilon}^i \Psi_\sigma^k) (\bar{\Psi}_\rho^j \gamma^m \gamma_* \gamma_m \gamma_\mu \Psi_\nu^l) \\
&\quad -\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_2^{kl} (\bar{\varepsilon}^i \gamma_* \Psi_\sigma^k) (\bar{\Psi}_\rho^j \gamma^m \gamma_* \gamma_m \gamma_\mu \Psi_\nu^l) \\
&\quad -\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_2^{kl} (\bar{\varepsilon}^i \gamma_a \Psi_\sigma^k) (\bar{\Psi}_\rho^j \gamma^m \gamma^a \gamma_* \gamma_m \gamma_\mu \Psi_\nu^l) \\
&\quad +\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_2^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_a \Psi_\sigma^k) (\bar{\Psi}_\rho^j \gamma^m \gamma_* \gamma^a \gamma_* \gamma_m \gamma_\mu \Psi_\nu^l) \\
&\quad +\frac{1}{8} \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_2^{kl} (\bar{\varepsilon}^i \gamma_{ab} \Psi_\sigma^k) (\bar{\Psi}_\rho^j \gamma^m \gamma^{ab} \gamma_* \gamma_m \gamma_\mu \Psi_\nu^l), \\
&= -\epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_2^{kl} (\bar{\varepsilon}^i \Psi_\mu^k) (\bar{\Psi}_\nu^j \gamma_* \gamma_\rho \Psi_\sigma^l) \\
&\quad +\epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_2^{kl} (\bar{\varepsilon}^i \gamma_* \Psi_\mu^k) (\bar{\Psi}_\nu^j \gamma_\rho \Psi_\sigma^l) \\
&\quad -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_2^{kl} (\bar{\varepsilon}^i \gamma_m \Psi_\mu^k) (\bar{\Psi}_\nu^k \gamma_* \gamma^m \gamma_\rho \Psi_\sigma^l) \\
&\quad -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_2^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_m \Psi_\mu^k) (\bar{\Psi}_\nu^j \gamma^m \gamma_\rho \Psi_\sigma^l) \\
&\quad +\frac{1}{8} \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_2^{kl} (\bar{\varepsilon}^i \gamma_{ab} \Psi_\mu^k) (\underline{\bar{\Psi}_\nu^j \gamma_* \gamma^m \gamma^{ab}} \underline{\gamma_m \gamma_\rho \Psi_\sigma^l}),
\end{aligned}$$

$$(1c) \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_2^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_m \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma^m \Psi_\sigma^k) = -\epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_2^{kl} (\bar{\varepsilon}^i \Psi_\mu^k) (\bar{\Psi}_\nu^l \gamma_* \gamma_\rho \Psi_\sigma^j) + \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_2^{kl} (\bar{\varepsilon}^i \gamma_* \Psi_\mu^k) (\bar{\Psi}_\nu^j \gamma_\rho \Psi_\sigma^l) \\ - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_2^{kl} (\bar{\varepsilon}^i \gamma_m \Psi_\mu^k) (\bar{\Psi}_\nu^j \gamma_* \gamma^m \gamma_\rho \Psi_\sigma^l) - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_2^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_m \Psi_\mu^k) (\bar{\Psi}_\nu^j \gamma^m \gamma_\rho \Psi_\sigma^l)$$

$$(1a) \Rightarrow \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_2^{kl} (\bar{\epsilon}^i \gamma_m \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \Psi_\sigma^k) - \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_2^{kl} (\bar{\epsilon}^i \gamma_* \gamma_m \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma^m \Psi_\sigma^k) = \\ 2\epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_2^{kl} (\bar{\epsilon}^i \gamma_m \Psi_\mu^l) (\bar{\Psi}_\nu^j \gamma_* \gamma^m \gamma_\rho \Psi_\sigma^l) - 2\epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_2^{kl} (\bar{\epsilon}^i \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma_* \Psi_\sigma^l) + 2\epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_2^{kl} (\bar{\epsilon}^i \gamma_* \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \Psi_\sigma^l).$$

$$(1b)-(1c) \quad \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_2^{kl} (\bar{\varepsilon}^i \gamma_m \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \Psi_\sigma^k) - \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_2^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_m \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma^m \Psi_\sigma^k) = \\ + 2\epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_2^{kl} (\bar{\varepsilon}^i \Psi_\mu^k) (\bar{\Psi}_\nu^j \gamma_* \gamma_\rho \Psi_\sigma^l) - 2\epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_2^{kl} (\bar{\varepsilon}^i \gamma_* \Psi_\mu^k) (\bar{\Psi}_\nu^j \gamma_\rho \Psi_\sigma^l)$$

$$\Rightarrow (1) \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_2^{kl} (\bar{\epsilon}^i \gamma_m \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_* \gamma^m \gamma_\rho \Psi_\sigma^l) = -\epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_2^{kl} (\bar{\epsilon}^i \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma_* \Psi_\sigma^k) + \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_2^{kl} (\bar{\epsilon}^i \gamma_* \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \Psi_\sigma^k) + \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_2^{kl} (\bar{\epsilon}^i \Psi_\mu^k) (\bar{\Psi}_\nu^j \gamma_* \gamma_\rho \Psi_\sigma^l) - \epsilon^{\mu\nu\rho\sigma} S_1^{ij} S_2^{kl} (\bar{\epsilon}^i \gamma_* \Psi_\mu^k) (\bar{\Psi}_\nu^j \gamma_\rho \Psi_\sigma^l)$$

Fierz 2a

$$(\bar{\chi}_1\chi_2)(\bar{\chi}_3\chi_4) = -\frac{1}{4}(\bar{\chi}_3\chi_2)(\bar{\chi}_1\chi_4) - \frac{1}{4}(\bar{\chi}_3\gamma_*\chi_2)(\bar{\chi}_1\gamma_*\chi_4)$$

$$\begin{aligned}
& -\frac{1}{4}(\bar{\chi}_3 \gamma_a \chi_2)(\bar{\chi}_1 \gamma^a \chi_4) + \frac{1}{4}(\bar{\chi}_3 \gamma_* \gamma_a \chi_2)(\bar{\chi}_1 \gamma_* \gamma^a \chi_4) \\
& + \frac{1}{8}(\bar{\chi}_3 \gamma_{ab} \chi_2)(\bar{\chi}_1 \gamma^{ab} \chi_4)
\end{aligned}$$

$$\bar{\chi}_1 = \bar{\Psi}_\rho^k \gamma_* \gamma^m, \chi_2 = \Psi_\sigma^l, \bar{\chi}_3 = \bar{\varepsilon}^i \gamma_\mu, \chi_4 = \gamma_m \Psi_\nu^j$$

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\Psi}_\rho^k \gamma_* \gamma^m \Psi_\sigma^l) (\bar{\varepsilon}^i \gamma_\mu \gamma_m \Psi_\nu^j) &= -\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_\mu \Psi_\sigma^l) (\bar{\Psi}_\rho^k \gamma_* \gamma^m \gamma_m \Psi_\nu^j) \\
&\quad -\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_\mu \gamma_* \Psi_\sigma^l) (\bar{\Psi}_\rho^k \gamma_* \gamma^m \gamma_* \gamma_m \Psi_\nu^j) \\
&\quad -\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_\mu \gamma_a \Psi_\sigma^l) (\bar{\Psi}_\rho^k \gamma_* \gamma^m \gamma^a \gamma_m \Psi_\nu^j) \\
&\quad +\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_\mu \gamma_* \gamma_a \Psi_\sigma^l) (\bar{\Psi}_\rho^k \gamma_* \gamma^m \gamma_* \gamma^a \gamma_m \Psi_\nu^j) \\
&\quad +\frac{1}{8} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_\mu \gamma_{ab} \Psi_\sigma^l) (\bar{\Psi}_\rho^k \gamma_* \gamma^m \gamma^{ab} \gamma_m \Psi_\nu^j), \\
&= +\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^j) \\
&\quad +\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \Psi_\sigma^j) \\
&\quad -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_\mu \gamma_m \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \gamma^m \Psi_\sigma^j) \\
&\quad +\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^j),
\end{aligned}$$

$$\begin{aligned}
(2a) \quad 0 &= \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \Psi_\sigma^j) + \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \Psi_\sigma^j) \\
&\quad - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_\mu \gamma_m \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \gamma^m \Psi_\sigma^j) + \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^j).
\end{aligned}$$

Fierz 2b

$$\begin{aligned}
(\bar{\chi}_1 \chi_2)(\bar{\chi}_3 \chi_4) &= -\frac{1}{4}(\bar{\chi}_3 \chi_2)(\bar{\chi}_1 \chi_4) - \frac{1}{4}(\bar{\chi}_3 \gamma_* \chi_2)(\bar{\chi}_1 \gamma_* \chi_4) \\
&\quad -\frac{1}{4}(\bar{\chi}_3 \gamma_a \chi_2)(\bar{\chi}_1 \gamma^a \chi_4) + \frac{1}{4}(\bar{\chi}_3 \gamma_* \gamma_a \chi_2)(\bar{\chi}_1 \gamma_* \gamma^a \chi_4) \\
&\quad +\frac{1}{8}(\bar{\chi}_3 \gamma_{ab} \chi_2)(\bar{\chi}_1 \gamma^{ab} \chi_4)
\end{aligned}$$

$$\bar{\chi}_1 = \bar{\Psi}_\rho^k \gamma^m, \chi_2 = \Psi_\sigma^l, \bar{\chi}_3 = \bar{\varepsilon}^i \gamma_\mu, \chi_4 = \gamma_m \Psi_\nu^j$$

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^l) (\bar{\varepsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^j) &= -\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_\mu \Psi_\sigma^l) (\bar{\Psi}_\rho^k \gamma^m \gamma_m \Psi_\nu^j) \\
&\quad -\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_\mu \gamma_* \Psi_\sigma^l) (\bar{\Psi}_\rho^k \gamma^m \gamma_* \gamma_m \Psi_\nu^j) \\
&\quad -\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_\mu \gamma_a \Psi_\sigma^l) (\bar{\Psi}_\rho^k \gamma^m \gamma^a \gamma_m \Psi_\nu^j) \\
&\quad +\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_\mu \gamma_* \gamma_a \Psi_\sigma^l) (\bar{\Psi}_\rho^k \gamma^m \gamma_* \gamma^a \gamma_m \Psi_\nu^j) \\
&\quad +\frac{1}{8} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_\mu \gamma_{ab} \Psi_\sigma^l) (\bar{\Psi}_\rho^k \gamma^m \gamma^{ab} \gamma_m \Psi_\nu^j), \\
&= +\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \Psi_\sigma^j) \\
&\quad +\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^j)
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^j) \\
& + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_\mu \gamma_m \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \gamma^m \Psi_\sigma^j),
\end{aligned}$$

$$(2b) \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^j) = +\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^j) + \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \Psi_\sigma^j)$$

$$+\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_\mu \gamma_m \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^j) - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^j)$$

$$(2a)+(2b) \Rightarrow (2) \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^j) (\bar{\Psi}_\rho^k \gamma^m \Psi_\sigma^l) = -2\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^k \gamma_* \Psi_\sigma^k) \\
-2\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \Psi_\sigma^k)$$

Fierz 3a

$$\begin{aligned}
(\bar{\chi}_1 \chi_2)(\bar{\chi}_3 \chi_4) &= -\frac{1}{4}(\bar{\chi}_3 \chi_2)(\bar{\chi}_1 \chi_4) - \frac{1}{4}(\bar{\chi}_3 \gamma_* \chi_2)(\bar{\chi}_1 \gamma_* \chi_4) \\
&\quad -\frac{1}{4}(\bar{\chi}_3 \gamma_a \chi_2)(\bar{\chi}_1 \gamma^a \chi_4) + \frac{1}{4}(\bar{\chi}_3 \gamma_* \gamma_a \chi_2)(\bar{\chi}_1 \gamma_* \gamma^a \chi_4) \\
&\quad +\frac{1}{8}(\bar{\chi}_3 \gamma_{ab} \chi_2)(\bar{\chi}_1 \gamma^{ab} \chi_4)
\end{aligned}$$

$$\bar{\chi}_1 = \bar{\Psi}_\nu^k \gamma_* \gamma_\rho, \chi_2 = \Psi_\sigma^l, \bar{\chi}_3 = \bar{\epsilon}^i, \chi_4 = \Psi_\mu^j$$

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \Psi_\sigma^l) (\bar{\epsilon}^i \Psi_\mu^j) &= -\frac{1}{4}(\bar{\epsilon}^i \Psi_\sigma^l) (\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \Psi_\mu^j) - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \Psi_\sigma^l) (\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \gamma_* \Psi_\mu^j) \\
&\quad -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_a \Psi_\sigma^l) (\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \gamma^a \Psi_\mu^j) \\
&\quad +\frac{1}{4}\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_a \Psi_\sigma^l) (\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \gamma_* \gamma^a \Psi_\mu^j) \\
&\quad +\frac{1}{8}\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_{ab} \Psi_\sigma^l) (\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \gamma^{ab} \Psi_\mu^j), \\
&= +\frac{1}{4}\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \Psi_\mu^l) (\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \Psi_\sigma^j) - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \Psi_\mu^l) (\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \Psi_\sigma^j) \\
&\quad +\frac{1}{4}\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_m \Psi_\mu^l) (\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \gamma^m \Psi_\sigma^j) \\
&\quad +\frac{1}{4}\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_m \Psi_\mu^l) (\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \gamma^m \Psi_\sigma^j) \\
&\quad -\frac{1}{8}\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_{mn} \Psi_\mu^l) (\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \gamma^{mn} \Psi_\sigma^j), \\
&\stackrel{F}{=} +\frac{1}{4}\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \Psi_\mu^l) (\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \Psi_\sigma^j) - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \Psi_\mu^l) (\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \Psi_\sigma^j) \\
&\quad -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_m \Psi_\mu^l) (\bar{\Psi}_\nu^j \gamma_* \gamma^m \gamma_\rho \Psi_\sigma^k) \\
&\quad -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_m \Psi_\mu^l) (\bar{\Psi}_\nu^j \gamma^m \gamma_\rho \Psi_\sigma^k) \\
&\quad -\frac{1}{8}\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_{mn} \Psi_\mu^l) (\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \gamma^{mn} \Psi_\sigma^j),
\end{aligned}$$

$$(3a) 0 = \frac{1}{4}\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \Psi_\mu^l) (\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \Psi_\sigma^j) - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \Psi_\mu^l) (\bar{\Psi}_\nu^k \gamma_\rho \Psi_\sigma^j) \\
-\frac{1}{4}\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_m \Psi_\mu^l) (\bar{\Psi}_\nu^j \gamma_* \gamma^m \gamma_\rho \Psi_\sigma^k) - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_m \Psi_\mu^l) (\bar{\Psi}_\nu^j \gamma^m \gamma_\rho \Psi_\sigma^k) \\
-\frac{1}{8}\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_{mn} \Psi_\mu^l) (\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \gamma^{mn} \Psi_\sigma^j)$$

Fierz 3b

$$\begin{aligned}
(\bar{\chi}_1 \chi_2)(\bar{\chi}_3 \chi_4) &= -\frac{1}{4}(\bar{\chi}_3 \chi_2)(\bar{\chi}_1 \chi_4) - \frac{1}{4}(\bar{\chi}_3 \gamma_* \chi_2)(\bar{\chi}_1 \gamma_* \chi_4) \\
&\quad - \frac{1}{4}(\bar{\chi}_3 \gamma_a \chi_2)(\bar{\chi}_1 \gamma^a \chi_4) + \frac{1}{4}(\bar{\chi}_3 \gamma_* \gamma_a \chi_2)(\bar{\chi}_1 \gamma_* \gamma^a \chi_4) \\
&\quad + \frac{1}{8}(\bar{\chi}_3 \gamma_{ab} \chi_2)(\bar{\chi}_1 \gamma^{ab} \chi_4)
\end{aligned}$$

$$\bar{\chi}_1 = \bar{\Psi}_\nu^k \gamma_\rho, \chi_2 = \Psi_\sigma^l, \bar{\chi}_3 = \bar{\epsilon}^i, \chi_4 = \gamma_* \Psi_\mu^j$$

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\Psi}_\nu^k \gamma_\rho \Psi_\sigma^l) (\bar{\epsilon}^i \gamma_* \Psi_\mu^j) &= -\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\epsilon}^i \Psi_\sigma^l) (\bar{\Psi}_\nu^k \gamma_\rho \gamma_* \Psi_\mu^j) - \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\epsilon}^i \gamma_a \Psi_\sigma^l) (\bar{\Psi}_\nu^k \gamma_\rho \gamma^a \gamma_* \Psi_\mu^j) \\
&\quad + \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_a \Psi_\sigma^l) (\bar{\Psi}_\nu^k \gamma_\rho \gamma_* \gamma^a \gamma_* \Psi_\mu^j) \\
&\quad + \frac{1}{8} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\epsilon}^i \gamma_{ab} \Psi_\sigma^l) (\bar{\Psi}_\nu^j \gamma_\rho \gamma^{ab} \gamma_* \Psi_\mu^i), \\
&= -\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\epsilon}^i \Psi_\mu^l) (\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \Psi_\sigma^j) + \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\epsilon}^i \gamma_m \Psi_\mu^l) (\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \gamma^m \Psi_\sigma^j) \\
&\quad + \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_m \Psi_\mu^l) (\bar{\Psi}_\nu^k \gamma_\rho \gamma^m \Psi_\sigma^j) \\
&\quad + \frac{1}{8} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\epsilon}^i \gamma_{mn} \Psi_\mu^l) (\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \gamma^{mn} \Psi_\sigma^j), \\
&\stackrel{F}{=} -\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\epsilon}^i \Psi_\mu^l) (\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \Psi_\sigma^j) + \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\epsilon}^i \gamma_* \Psi_\mu^l) (\bar{\Psi}_\nu^k \gamma_\rho \Psi_\sigma^j) \\
&\quad - \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\epsilon}^i \gamma_m \Psi_\mu^l) (\bar{\Psi}_\nu^j \gamma_* \gamma^m \gamma_\rho \Psi_\sigma^k) \\
&\quad - \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_m \Psi_\mu^l) (\bar{\Psi}_\nu^j \gamma^m \gamma_\rho \Psi_\sigma^k) \\
&\quad + \frac{1}{8} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\epsilon}^i \gamma_{mn} \Psi_\mu^l) (\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \gamma^{mn} \Psi_\sigma^j),
\end{aligned}$$

$$\begin{aligned}
(3b) \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\epsilon}^i \gamma_* \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_\rho \Psi_\sigma^l) &= -\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\epsilon}^i \Psi_\mu^l) (\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \Psi_\sigma^j) + \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\epsilon}^i \gamma_* \Psi_\mu^l) (\bar{\Psi}_\nu^k \gamma_\rho \Psi_\sigma^j) \\
&\quad - \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\epsilon}^i \gamma_m \Psi_\mu^l) (\bar{\Psi}_\nu^j \gamma_* \gamma^m \gamma_\rho \Psi_\sigma^k) - \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_m \Psi_\mu^l) (\bar{\Psi}_\nu^j \gamma^m \gamma_\rho \Psi_\sigma^k) \\
&\quad + \frac{1}{8} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\epsilon}^i \gamma_{mn} \Psi_\mu^l) (\bar{\Psi}_\nu^k \gamma_* \gamma_\rho \gamma^{mn} \Psi_\sigma^j)
\end{aligned}$$

$$\begin{aligned}
(3a) + (3b) \Rightarrow (3) \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\epsilon}^i \gamma_* \Psi_\mu^j) (\bar{\Psi}_\nu^k \gamma_\rho \Psi_\sigma^l) &= -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\epsilon}^i \gamma_m \Psi_\mu^l) (\bar{\Psi}_\nu^j \gamma_* \gamma^m \gamma_\rho \Psi_\sigma^k) \\
&\quad - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_m \Psi_\mu^l) (\bar{\Psi}_\nu^j \gamma^m \gamma_\rho \Psi_\sigma^k)
\end{aligned}$$

Fierz 4a

$$\begin{aligned}
(\bar{\chi}_1 \chi_2)(\bar{\chi}_3 \chi_4) &= -\frac{1}{4}(\bar{\chi}_3 \chi_2)(\bar{\chi}_1 \chi_4) - \frac{1}{4}(\bar{\chi}_3 \gamma_* \chi_2)(\bar{\chi}_1 \gamma_* \chi_4) \\
&\quad - \frac{1}{4}(\bar{\chi}_3 \gamma_a \chi_2)(\bar{\chi}_1 \gamma^a \chi_4) + \frac{1}{4}(\bar{\chi}_3 \gamma_* \gamma_a \chi_2)(\bar{\chi}_1 \gamma_* \gamma^a \chi_4) \\
&\quad + \frac{1}{8}(\bar{\chi}_3 \gamma_{ab} \chi_2)(\bar{\chi}_1 \gamma^{ab} \chi_4)
\end{aligned}$$

$$\bar{\chi}_1 = \bar{\Psi}_\nu^j \gamma_* \gamma^m, \chi_2 = \gamma_\rho \Psi_\sigma^k, \bar{\chi}_3 = \bar{\varepsilon}^i, \chi_4 = \gamma_m \Psi_\mu^l$$

$$\begin{aligned} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\Psi}_\nu^j \gamma_* \gamma^m \gamma_\rho \Psi_\sigma^k) (\bar{\varepsilon}^i \gamma_m \Psi_\mu^l) &= -\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\varepsilon}^i \gamma_\rho \Psi_\sigma^k) (\bar{\Psi}_\nu^j \gamma_* \gamma^m \gamma_m \Psi_\mu^l) \\ &\quad -\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_\rho \Psi_\sigma^k) (\bar{\Psi}_\nu^j \gamma_* \gamma^m \gamma_m \Psi_\mu^l) \\ &\quad -\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\varepsilon}^i \gamma_a \gamma_\rho \Psi_\sigma^k) (\bar{\Psi}_\nu^j \gamma_* \gamma^m \gamma^a \gamma_m \Psi_\mu^l) \\ &\quad +\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_a \gamma_\rho \Psi_\sigma^k) (\bar{\Psi}_\nu^j \gamma_* \gamma^m \gamma_* \gamma^a \gamma_m \Psi_\mu^l) \\ &\quad +\frac{1}{8} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\varepsilon}^i \gamma_{ab} \gamma_\rho \Psi_\sigma^k) (\bar{\Psi}_\nu^j \gamma_* \gamma^m \gamma^{ab} \gamma_m \Psi_\mu^l), \\ &= +\epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\varepsilon}^i \gamma_\mu \Psi_\nu^k) (\bar{\Psi}_\rho^j \gamma_* \Psi_\sigma^l) \\ &\quad -\epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_\mu \Psi_\nu^k) (\bar{\Psi}_\rho^j \Psi_\sigma^l) \\ &\quad -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\varepsilon}^i \gamma_m \gamma_\mu \Psi_\nu^k) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \Psi_\sigma^l) \\ &\quad -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_m \gamma_\mu \Psi_\nu^k) (\bar{\Psi}_\rho^j \gamma^m \Psi_\sigma^l), \end{aligned}$$

$$(4a) \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\varepsilon}^i \gamma_m \Psi_\mu^l) (\bar{\Psi}_\nu^j \gamma_* \gamma^m \gamma_\rho \Psi_\sigma^k) = \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\varepsilon}^i \gamma_\mu \Psi_\nu^k) (\bar{\Psi}_\rho^j \gamma_* \Psi_\sigma^l) - \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_\mu \Psi_\nu^k) (\bar{\Psi}_\rho^j \gamma_* \gamma_\mu \Psi_\nu^l) \\ - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\varepsilon}^i \gamma_m \gamma_\mu \Psi_\nu^k) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \Psi_\sigma^l) - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_m \gamma_\mu \Psi_\nu^k) (\bar{\Psi}_\rho^j \gamma^m \Psi_\sigma^l)$$

Fierz 4b

$$\begin{aligned} (\bar{\chi}_1 \chi_2) (\bar{\chi}_3 \chi_4) &= -\frac{1}{4} (\bar{\chi}_3 \chi_2) (\bar{\chi}_1 \chi_4) - \frac{1}{4} (\bar{\chi}_3 \gamma_* \chi_2) (\bar{\chi}_1 \gamma_* \chi_4) \\ &\quad -\frac{1}{4} (\bar{\chi}_3 \gamma_a \chi_2) (\bar{\chi}_1 \gamma^a \chi_4) + \frac{1}{4} (\bar{\chi}_3 \gamma_* \gamma_a \chi_2) (\bar{\chi}_1 \gamma_* \gamma^a \chi_4) \\ &\quad +\frac{1}{8} (\bar{\chi}_3 \gamma_{ab} \chi_2) (\bar{\chi}_1 \gamma^{ab} \chi_4) \end{aligned}$$

$$\bar{\chi}_1 = \bar{\Psi}_\nu^j \gamma^m, \chi_2 = \gamma_\rho \Psi_\sigma^k, \bar{\chi}_3 = \bar{\varepsilon}^i, \chi_4 = \gamma_* \gamma_m \Psi_\mu^l$$

$$\begin{aligned} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\Psi}_\nu^j \gamma^m \gamma_\rho \Psi_\sigma^k) (\bar{\varepsilon}^i \gamma_* \gamma_m \Psi_\mu^l) &= -\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\varepsilon}^i \gamma_\rho \Psi_\sigma^k) (\bar{\Psi}_\nu^j \gamma^m \gamma_* \gamma_m \Psi_\mu^l) \\ &\quad -\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_\rho \Psi_\sigma^k) (\bar{\Psi}_\nu^j \gamma^m \gamma_* \gamma_* \gamma_m \Psi_\mu^l) \\ &\quad -\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\varepsilon}^i \gamma_a \gamma_\rho \Psi_\sigma^k) (\bar{\Psi}_\nu^j \gamma^m \gamma^a \gamma_* \gamma_m \Psi_\mu^l) \\ &\quad +\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_a \gamma_\rho \Psi_\sigma^k) (\bar{\Psi}_\nu^j \gamma^m \gamma_* \gamma^a \gamma_* \gamma_m \Psi_\mu^l) \\ &\quad +\frac{1}{8} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\varepsilon}^i \gamma_{ab} \gamma_\rho \Psi_\sigma^k) (\bar{\Psi}_\nu^j \gamma^m \gamma^{ab} \gamma_* \gamma_m \Psi_\mu^l), \\ &= -\epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\varepsilon}^i \gamma_\mu \Psi_\nu^k) (\bar{\Psi}_\rho^j \gamma_* \Psi_\sigma^l) \\ &\quad +\epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_\mu \Psi_\nu^k) (\bar{\Psi}_\rho^j \Psi_\sigma^l) \\ &\quad -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\varepsilon}^i \gamma_m \gamma_\mu \Psi_\nu^k) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \Psi_\sigma^l) \\ &\quad -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} S_2^{ij} S_1^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_m \gamma_\mu \Psi_\nu^k) (\bar{\Psi}_\rho^j \gamma^m \Psi_\sigma^l) \end{aligned}$$

$$+ \frac{1}{8} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_{ab} \gamma_\mu \Psi_\nu^k) \underline{(\bar{\Psi}_\rho^j \gamma_* \gamma^m \gamma^{ab} \gamma_m \Psi_\sigma^l)},$$

$$(4b) \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_m \Psi_\mu^l) (\bar{\Psi}_\nu^j \gamma^m \gamma_\rho \Psi_\sigma^k) = -\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_\mu \Psi_\nu^k) (\bar{\Psi}_\rho^j \gamma_* \Psi_\sigma^l) + \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_\mu \Psi_\nu^k) (\bar{\Psi}_\rho^j \Psi_\sigma^l)$$

$$-\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_m \gamma_\mu \Psi_\nu^k) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \Psi_\sigma^l) - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_m \gamma_\mu \Psi_\nu^k) (\bar{\Psi}_\rho^j \gamma^m \Psi_\sigma^l)$$

$$(4a)+(4b) \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_m \Psi_\mu^l) (\bar{\Psi}_\nu^j \gamma_* \gamma^m \gamma_\rho \Psi_\sigma^k) + \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_m \Psi_\mu^l) (\bar{\Psi}_\nu^j \gamma^m \gamma_\rho \Psi_\sigma^k) =$$

$$-\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_m \gamma_\mu \Psi_\nu^k) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \Psi_\sigma^l) - \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_m \gamma_\mu \Psi_\nu^k) (\bar{\Psi}_\rho^j \gamma^m \Psi_\sigma^l)$$

$$\gamma_m \gamma_\mu = 2e_m - \gamma_\mu \gamma_m \Rightarrow (4) \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_m \Psi_\mu^l) (\bar{\Psi}_\nu^j \gamma_* \gamma^m \gamma_\rho \Psi_\sigma^k) + \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_m \Psi_\mu^l) (\bar{\Psi}_\nu^j \gamma^m \gamma_\rho \Psi_\sigma^k) =$$

$$-2\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \Psi_\mu^k) (\bar{\Psi}_\nu^j \gamma_* \gamma_\rho \Psi_\sigma^l) - 2\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \Psi_\mu^k) (\bar{\Psi}_\nu^j \gamma_\rho \Psi_\sigma^l)$$

$$+\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_\mu \gamma_m \Psi_\nu^k) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \Psi_\sigma^l) + \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^k) (\bar{\Psi}_\rho^j \gamma^m \Psi_\sigma^l)$$

Fierz 5a

$$(\bar{\chi}_1 \chi_2)(\bar{\chi}_3 \chi_4) = -\frac{1}{4} (\bar{\chi}_3 \chi_2)(\bar{\chi}_1 \chi_4) - \frac{1}{4} (\bar{\chi}_3 \gamma_* \chi_2)(\bar{\chi}_1 \gamma_* \chi_4)$$

$$-\frac{1}{4} (\bar{\chi}_3 \gamma_a \chi_2)(\bar{\chi}_1 \gamma^a \chi_4) + \frac{1}{4} (\bar{\chi}_3 \gamma_* \gamma_a \chi_2)(\bar{\chi}_1 \gamma_* \gamma^a \chi_4)$$

$$+\frac{1}{8} (\bar{\chi}_3 \gamma_{ab} \chi_2)(\bar{\chi}_1 \gamma^{ab} \chi_4)$$

$$\bar{\chi}_1 = \bar{\Psi}_\rho^j \gamma_* \gamma^m, \chi_2 = \Psi_\sigma^l, \bar{\chi}_3 = \bar{\epsilon}^i \gamma_\mu, \chi_4 = \gamma_m \Psi_\nu^k$$

$$\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\Psi}_\rho^j \gamma_* \gamma^m \Psi_\sigma^l) (\bar{\epsilon}^i \gamma_\mu \gamma_m \Psi_\nu^k) = -\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_\mu \Psi_\sigma^l) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \gamma_m \Psi_\nu^k)$$

$$-\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_\mu \gamma_* \Psi_\sigma^l) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \gamma_* \gamma_m \Psi_\nu^k)$$

$$-\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_\mu \gamma_a \Psi_\sigma^l) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \gamma^a \gamma_m \Psi_\nu^k)$$

$$+\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_\mu \gamma_* \gamma_a \Psi_\sigma^l) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \gamma_* \gamma^a \gamma_m \Psi_\nu^k)$$

$$+\frac{1}{8} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_\mu \gamma_{ab} \Psi_\sigma^l) \underline{(\bar{\Psi}_\rho^j \gamma_* \gamma^m \gamma^{ab} \gamma_m \Psi_\nu^k)},$$

$$= +\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma_* \Psi_\sigma^k)$$

$$+\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \Psi_\sigma^k)$$

$$-\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_\mu \gamma_m \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \Psi_\sigma^k)$$

$$+\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma^m \Psi_\sigma^k),$$

$$(5a) \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_\mu \gamma_m \Psi_\nu^k) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \Psi_\sigma^l) = \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma_* \Psi_\sigma^k) + \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma_* \gamma_\mu \Psi_\sigma^l)$$

$$-\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_\mu \gamma_m \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \Psi_\sigma^k) + \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\epsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma^m \Psi_\sigma^k)$$

Fierz 5b

$$(\bar{\chi}_1 \chi_2)(\bar{\chi}_3 \chi_4) = -\frac{1}{4} (\bar{\chi}_3 \chi_2)(\bar{\chi}_1 \chi_4) - \frac{1}{4} (\bar{\chi}_3 \gamma_* \chi_2)(\bar{\chi}_1 \gamma_* \chi_4)$$

$$\begin{aligned}
& -\frac{1}{4}(\bar{\chi}_3 \gamma_a \chi_2)(\bar{\chi}_1 \gamma^a \chi_4) + \frac{1}{4}(\bar{\chi}_3 \gamma_* \gamma_a \chi_2)(\bar{\chi}_1 \gamma_* \gamma^a \chi_4) \\
& + \frac{1}{8}(\bar{\chi}_3 \gamma_{ab} \chi_2)(\bar{\chi}_1 \gamma^{ab} \chi_4)
\end{aligned}$$

$$\bar{\chi}_1 = \bar{\Psi}_\rho^j \gamma^m, \chi_2 = \Psi_\sigma^l, \bar{\chi}_3 = \bar{\varepsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^k, \chi_4 = \gamma_m \Psi_\nu^k$$

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\Psi}_\rho^j \gamma^m \Psi_\sigma^l) (\bar{\varepsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^k) &= -\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_\mu \Psi_\sigma^l) (\bar{\Psi}_\rho^k j \gamma^m \gamma_m \Psi_\nu^k) \\
&\quad -\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_\mu \gamma_* \Psi_\sigma^l) (\bar{\Psi}_\rho^j \gamma^m \gamma_* \gamma_m \Psi_\nu^k) \\
&\quad -\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_\mu \gamma_a \Psi_\sigma^l) (\bar{\Psi}_\rho^j \gamma^m \gamma^a \gamma_m \Psi_\nu^k) \\
&\quad +\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_\mu \gamma_* \gamma_a \Psi_\sigma^l) (\bar{\Psi}_\rho^j \gamma^m \gamma_* \gamma^a \gamma_m \Psi_\nu^k) \\
&\quad +\frac{1}{8} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_\mu \gamma_{ab} \Psi_\sigma^l) (\bar{\Psi}_\rho^j \gamma^m \gamma^{ab} \gamma_m \Psi_\nu^k), \\
&= +\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \Psi_\sigma^k) \\
&\quad +\epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma_* \Psi_\sigma^k) \\
&\quad -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma^m \Psi_\sigma^k) \\
&\quad +\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_\mu \gamma_m \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \Psi_\sigma^k),
\end{aligned}$$

$$(5b) \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma^m \Psi_\sigma^l) = \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma_* \Psi_\sigma^k) + \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma^m \Psi_\sigma^k) + \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_\mu \gamma_m \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \Psi_\sigma^k) - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma^m \Psi_\sigma^k)$$

$$(5a)+(5b) \Rightarrow (5) \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_\mu \gamma_m \Psi_\nu^k) (\bar{\Psi}_\rho^j \gamma_* \gamma^m \Psi_\sigma^l) + \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_\mu \gamma_m \Psi_\nu^k) (\bar{\Psi}_\rho^j \gamma^m \Psi_\sigma^l) = + 2 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \gamma_* \Psi_\sigma^k) + 2 \epsilon^{\mu\nu\rho\sigma} \mathbf{S}_2^{ij} \mathbf{S}_1^{kl} (\bar{\varepsilon}^i \gamma_* \gamma_\mu \Psi_\nu^l) (\bar{\Psi}_\rho^j \Psi_\sigma^k)$$

E Miscellaneous

E.1 Calculation of $D_{[\mu} D_{\nu]} \Psi_\rho^i$ and $D_{[\mu} D_{\nu]} \gamma_\rho^i$

Let's first calculate $D_{[\mu} D_{\nu]} \Psi_\rho^i$

$$\begin{aligned}
D_{[\mu} D_{\nu]} \Psi_\rho^i &= D_\mu D_\nu \Psi_\rho^i - D_\nu D_\mu \Psi_\rho^i, \\
&\stackrel{(a)}{=} D_\mu (\partial_\nu \chi + \frac{1}{4} \omega_{\nu ab} \gamma^{ab} \chi) - D_\nu (\partial_\mu \chi + \frac{1}{4} \omega_{\mu ab} \gamma^{ab} \chi), \\
&\stackrel{(b)}{=} \partial_\mu (\partial_\nu \chi + \frac{1}{4} \omega_{\nu ab} \gamma^{ab} \chi) + \frac{1}{4} \omega_{\mu cd} \gamma^{cd} (\partial_\nu \chi + \frac{1}{4} \omega_{\nu ab} \gamma^{ab} \chi) \\
&\quad - \partial_\nu (\partial_\mu \chi - \frac{1}{4} \omega_{\mu ab} \gamma^{ab} \chi) - \frac{1}{4} \omega_{\nu cd} \gamma^{cd} (\partial_\mu \chi - \frac{1}{4} \omega_{\mu ab} \gamma^{ab} \chi), \\
&= \cancel{\partial_\mu \partial_\nu \chi}^a + \frac{1}{4} \partial_\mu \omega_{\nu ab} \gamma^{ab} \chi + \frac{1}{4} \cancel{\omega_{\nu ab} \gamma^{ab}}^b \cancel{\partial_\mu \chi}^b + \frac{1}{4} \cancel{\omega_{\mu cd} \gamma^{cd}}^c \cancel{\partial_\nu \chi}^c + \frac{1}{16} \omega_{\mu cd} \omega_{\nu ab} \gamma^{cd} \gamma^{ab} \chi \\
&\quad - \cancel{\partial_\nu \partial_\mu \chi}^a - \frac{1}{4} \partial_\nu \omega_{\mu ab} \gamma^{ab} \chi - \frac{1}{4} \cancel{\omega_{\mu ab} \gamma^{ab}}^b \cancel{\partial_\nu \chi}^b - \frac{1}{4} \cancel{\omega_{\nu cd} \gamma^{cd}}^b \cancel{\partial_\mu \chi}^b - \frac{1}{16} \omega_{\nu cd} \omega_{\mu ab} \gamma^{cd} \gamma^{ab} \chi, \\
&= \frac{1}{4} \partial_\mu \omega_{\nu ab} \gamma^{ab} \chi - \frac{1}{4} \partial_\nu \omega_{\mu ab} \gamma^{ab} \chi + \frac{1}{16} \omega_{\mu ab} \omega_{\nu cd} \gamma^{ab} \gamma^{cd} \chi - \frac{1}{16} \omega_{\mu ab} \omega_{\nu cd} \gamma^{cd} \gamma^{ab} \chi
\end{aligned}$$

$$\begin{aligned}
&\stackrel{(c)}{=} \frac{1}{4} \partial_\mu \omega_{\nu ab} \gamma^{ab} \chi - \frac{1}{4} \partial_\nu \omega_{\mu ab} \gamma^{ab} \chi + \frac{1}{16} \omega_{\mu ab} \omega_{\nu cd} \gamma^{ab} \gamma^{cd} \chi - \frac{1}{16} \omega_{\mu ab} \omega_{\nu cd} \gamma^{cd} \gamma^{ab} \chi \\
&\stackrel{(d)}{=} \frac{1}{4} \partial_\mu \omega_{\nu ab} \gamma^{ab} \chi - \frac{1}{4} \partial_\nu \omega_{\mu ab} \gamma^{ab} \chi + \frac{1}{4} \omega_{\mu ba} \omega_{\nu cd} \eta^{ac} \gamma^{bd} - \frac{1}{4} \omega_{\nu ba} \omega_{\mu dc} \eta^{ad} \gamma^{bc} \chi \\
&\stackrel{(e)}{=} \frac{1}{4} R_{\mu\nu}^{mn} \gamma_{mn} \Psi_\rho^i.
\end{aligned}$$

The steps (a) and (b) use the definition (13).

The step (c) uses the relation $\gamma^{ab} \gamma^{cd} - \gamma^{cd} \gamma^{ab} = -2\eta^{ac} \gamma^{bd} + 2\eta^{bc} \gamma^{ad} + 2\eta^{ad} \gamma^{bc} - 2\eta^{bd} \gamma^{ac}$.

The step (d) uses the property $\omega_{\mu ab} = -\omega_{\mu ba}$.

The step (e) uses the definition (24).

Let's than calculate $D_{[\mu} D_{\nu]} \gamma_\rho$:

$$\begin{aligned}
D_{[\mu} D_{\nu]} \gamma_\rho &= D_{[\mu} D_{\nu]} (e_\rho^r \gamma_r), \\
&= D_{[\mu} D_{\nu]} e_\rho^r \gamma_r, \\
&= [D_\mu D_\nu e_\rho^r - D_\nu D_\mu e_\rho^r] \gamma_r, \\
&\stackrel{(a)}{=} [D_\mu (\partial_\nu e_\rho^r + \omega_\nu^r{}_k e_\rho^k) - D_\nu (\partial_\mu e_\rho^r + \omega_\mu^r{}_k e_\rho^k)] \gamma_r, \\
&\stackrel{(b)}{=} [\partial_\mu (\partial_\nu e_\rho^r + \omega_\nu^r{}_k e_\rho^k) + \omega_\mu^r{}_l (\partial_\nu e_\rho^l + \omega_\nu^r{}_l e_\rho^k) \\
&\quad - \partial_\nu (\partial_\mu e_\rho^r + \omega_\mu^r{}_k e_\rho^k) - \omega_\nu^r{}_l (\partial_\mu e_\rho^l + \omega_\mu^r{}_l e_\rho^k)] \gamma_r, \\
&= (\cancel{\partial_\mu \partial_\nu e_\rho^r}^a + \partial_\mu \omega_\nu^r{}_k e_\rho^k + \cancel{\omega_\nu^r{}_k \partial_\mu e_\rho^k}^b + \cancel{\omega_\mu^r{}_l \partial_\nu e_\rho^l}^c + \omega_\nu^r{}_l \omega_\mu^r{}_l e_\rho^k \\
&\quad - \cancel{\partial_\nu \partial_\mu e_\rho^r}^a - \partial_\nu \omega_\mu^r{}_k e_\rho^k - \cancel{\omega_\mu^r{}_k \partial_\nu e_\rho^k}^c - \cancel{\omega_\nu^r{}_l \partial_\mu e_\rho^l}^b - \omega_\nu^r{}_l \omega_\mu^r{}_l e_\rho^k) \gamma_r, \\
&\stackrel{(c)}{=} R_{\mu\nu}^{mn} \gamma_m e_{n\rho}.
\end{aligned}$$

The steps (a) and (b) use the definition (12).

The step (c) uses the definition (24).

E.2 Calculation of $[\tilde{D}_\mu, \tilde{D}_\nu] \varepsilon^i$

$$\begin{aligned}
[\tilde{D}_\mu, \tilde{D}_\nu] \varepsilon^i &= \tilde{D}_\mu \tilde{D}_\nu \varepsilon^i - \tilde{D}_\nu \tilde{D}_\mu \varepsilon^i, \\
&\stackrel{(a)}{=} \tilde{D}_\mu (D_\nu \varepsilon^i + \frac{\sqrt{\Lambda}}{2\sqrt{3}} M_1^{ij} \gamma_\nu \varepsilon^j) - \tilde{D}_\nu (D_\mu \varepsilon^i + \frac{\sqrt{\Lambda}}{2\sqrt{3}} M_1^{ij} \gamma_\mu \varepsilon^j), \\
&\stackrel{(b)}{=} D_\mu (D_\nu \varepsilon^i + \frac{\sqrt{\Lambda}}{2\sqrt{3}} M_1^{ij} \gamma_\nu \varepsilon^j) + \frac{\sqrt{\Lambda}}{2\sqrt{3}} M_1^{ik} \gamma_\mu (D_\nu \varepsilon^k + \frac{\sqrt{\Lambda}}{2\sqrt{3}} M_1^{kj} \gamma_\nu \varepsilon^j) \\
&\quad - D_\nu (D_\mu \varepsilon^i + \frac{\sqrt{\Lambda}}{2\sqrt{3}} M_1^{ij} \gamma_\mu \varepsilon^j) - \frac{\sqrt{\Lambda}}{2\sqrt{3}} M_1^{ik} \gamma_\nu (D_\mu \varepsilon^k + \frac{\sqrt{\Lambda}}{2\sqrt{3}} M_1^{kj} \gamma_\mu \varepsilon^j), \\
&\stackrel{(c)}{=} D_\mu D_\nu \varepsilon^i + \frac{\sqrt{\Lambda}}{2\sqrt{3}} M_1^{ij} D_\mu e_\nu^r \gamma_r \varepsilon^j + \cancel{\frac{\sqrt{\Lambda}}{2\sqrt{3}} M_1^{ij} \gamma_\nu D_\mu \varepsilon^j}^a + \cancel{\frac{\sqrt{\Lambda}}{2\sqrt{3}} M_1^{ik} \gamma_\mu D_\nu \varepsilon^k}^b + \frac{\Lambda}{12} M_1^{ik} M_1^{kj} \gamma_\mu \gamma_\nu \varepsilon^j \\
&\quad - D_\nu D_\mu \varepsilon^i - \cancel{\frac{\sqrt{\Lambda}}{2\sqrt{3}} M_1^{ij} D_\nu e_\mu^r \gamma_r \varepsilon^j}^a - \cancel{\frac{\sqrt{\Lambda}}{2\sqrt{3}} M_1^{ij} \gamma_\mu D_\nu \varepsilon^j}^b - \cancel{\frac{\sqrt{\Lambda}}{2\sqrt{3}} M_1^{ik} \gamma_\nu D_\mu \varepsilon^k}^a - \frac{\Lambda}{12} M_1^{ik} M_1^{kj} \gamma_\nu \gamma_\mu \varepsilon^j,
\end{aligned}$$

$$\begin{aligned}
&\stackrel{(d)}{=} [D_\mu, D_\nu] \varepsilon^i + \frac{\sqrt{\Lambda}}{4\sqrt{3}} \overline{\boldsymbol{M}_1^{ij} D_{[\mu} e_{\nu]}^r \gamma_r} \varepsilon^j \pm \frac{\Lambda}{12} \gamma_\mu \gamma_\nu \varepsilon^i \mp \frac{\Lambda}{12} \gamma_\nu \gamma_\mu \varepsilon^i, \\
&\stackrel{(e)}{=} \left(\frac{1}{4} R_{\mu\nu}{}^{mn} \gamma_{mn} \pm \frac{\Lambda}{6} \gamma_{\mu\nu} \right) \varepsilon^i.
\end{aligned}$$

The steps (a) and (b) use the definition (49).

The step(c) takes into account $D_\nu \gamma_\nu = D_\nu (e_\nu^r \gamma_r) = D_\nu e_\nu^r \gamma_r$.

The step (d) takes into account $(\boldsymbol{M}_1)^2 = \pm \mathbf{I}$ from (37) and $D_{[\rho} e_{\sigma]}^s = 0$ obtained from (31) when $\Psi_\mu^i = 0$.

The step (e) takes into account the first result obtained in Appendix E.1 and the relation $\gamma_{\mu\nu} = \frac{1}{2} \gamma_\mu \gamma_\nu - \frac{1}{2} \gamma_\nu \gamma_\mu$.

E.3 Lorentz transformation of the spin connection $\omega_\mu{}^{mn}$

From (12), let's calculate $\delta_L(D_\mu V^m)$ that must transform by (8) as a vector, that is $\delta_L(D_\mu V^m) \equiv -\lambda^m{}_k D_\mu V^k$

$$\begin{aligned}
\delta_L(D_\mu V^m) &= \delta_L(\partial_\mu V^m + \omega_\mu{}^m{}_k V^k), \\
&= \partial_\mu \delta_L V^m + \delta_L \omega_\mu{}^m{}_k V^k + \omega_\mu{}^m{}_k \delta_L V^k, \\
&= \partial_\mu(-\lambda^m{}_k V^k) + \delta_L \omega_\mu{}^m{}_k V^k + \omega_\mu{}^m{}_k(-\lambda^k{}_l V^l), \\
&= -\partial_\mu \lambda^m{}_k V^k - \lambda^m{}_k \partial_\mu V^k + \delta_L \omega_\mu{}^m{}_k V^k - \lambda^k{}_l \omega_\mu{}^m{}_k V^l, \\
&= -\lambda^m{}_k(\partial_\mu V^k + \omega_\mu{}^k{}_l V^l) + \lambda^m{}_k \omega_\mu{}^k{}_l V^l - \partial_\mu \lambda^m{}_k V^k + \delta_L \omega_\mu{}^m{}_k V^k - \lambda^k{}_l \omega_\mu{}^m{}_k V^l, \\
&= -\lambda^m{}_k D_\mu V^k + (\delta_L \omega_\mu{}^m{}_k - \partial_\mu \lambda^m{}_k) V^k - \lambda^l{}_k \omega_\mu{}^m{}_l + \lambda^m{}_l \omega_\mu{}^l{}_k) V^k,
\end{aligned}$$

which means with $\omega_\mu{}^{mn} = -\omega_\mu{}^{nm}$ and $\lambda_{ab} = -\lambda_{ba}$ that

$$\delta_L \omega_\mu{}^{mn} = \partial_\mu \lambda^{mn} - \lambda^{mk} \omega_{\mu k}{}^n + \lambda^{nk} \omega_{\mu k}{}^m.$$

Let's prove that this result is compatible with (13) that must transform by (9) as a spinor, that is $\delta_L(D_\mu \chi) \equiv -\frac{1}{4} \lambda_{ab} \gamma^{ab} D_\mu \chi$

$$\begin{aligned}
\delta_L(D_\mu \chi) &= \delta_L(\partial_\mu \chi + \frac{1}{4} \omega_{\mu ab} \gamma^{ab} \chi), \\
&= \partial_\mu \delta_L \chi + \frac{1}{4} \delta_L \omega_{\mu ab} \gamma^{ab} \chi + \frac{1}{4} \omega_{\mu ab} \gamma^{ab} \delta_L \chi, \\
&= \partial_\mu(-\frac{1}{4} \lambda_{ab} \gamma^{ab} \chi) + \frac{1}{4} (\partial_\mu \lambda_{ab} - \lambda_a{}^k \omega_{\mu kb} + \lambda_b{}^k \omega_{\mu ka}) \gamma^{ab} \chi + \frac{1}{4} \omega_{\mu ab} \gamma^{ab} (-\frac{1}{4} \lambda_{cd} \gamma^{cd} \chi), \\
&= -\frac{1}{4} \partial_\mu \lambda_{ab} \gamma^{ab} \chi - \frac{1}{4} \lambda_{ab} \gamma^{ab} \partial_\mu \chi + \frac{1}{4} \partial_\mu \lambda_{ab} \gamma^{ab} \chi - \frac{1}{4} \lambda_a{}^k \omega_{\mu kb} \gamma^{ab} \chi + \frac{1}{4} \lambda_b{}^k \omega_{\mu ka} \gamma^{ab} \chi \\
&\quad - \frac{1}{16} \lambda_{cd} \omega_{\mu ab} \gamma^{ab} \gamma^{cd} \chi, \\
&\stackrel{(a)}{=} -\frac{1}{4} \lambda_{ab} \gamma^{ab} \partial_\mu \chi - \frac{1}{4} \lambda_a{}^k \omega_{\mu kb} \gamma^{ab} \chi + \frac{1}{4} \lambda_b{}^k \omega_{\mu ka} \gamma^{ab} \chi - \frac{1}{16} \lambda_{cd} \omega_{\mu ab} \gamma^{cd} \gamma^{ab} \chi \\
&\quad + \frac{1}{4} \lambda_{cd} \omega_{\mu ab} \eta^{ac} \gamma^{bd} \chi - \frac{1}{4} \lambda_{cd} \omega_{\mu ab} \eta^{ad} \gamma^{bc} \chi, \\
&\stackrel{(b)}{=} -\frac{1}{4} \lambda_{ab} \gamma^{ab} \partial_\mu \chi - \frac{1}{4} \lambda_a{}^k \omega_{\mu kb} \gamma^{ab} \chi - \frac{1}{4} \lambda_b{}^k \omega_{\mu ka} \gamma^{ab} \chi - \frac{1}{16} \lambda_{ab} \gamma^{ab} \omega_{\mu cd} \gamma^{cd} \chi \\
&\quad + \frac{1}{4} \lambda_a{}^k \omega_{\mu kb} \gamma^{ab} \chi - \frac{1}{4} \lambda_b{}^k \omega_{\mu ka} \gamma^{ab} \chi,
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{4}\lambda_{ab}\gamma^{ab}(\partial_\mu\chi + \frac{1}{4}\omega_{\mu cd}\gamma^{cd}\chi) \\
&\stackrel{!}{=} -\frac{1}{4}\lambda_{ab}\gamma^{ab}D_\mu\chi,
\end{aligned}$$

The step (a) uses the relation $\gamma^{ab}\gamma^{cd} = \gamma^{cd}\gamma^{ab} - 2\eta^{ac}\gamma^{bd} + 2\eta^{bc}\gamma^{ad} + 2\eta^{ad}\gamma^{bc} - 2\eta^{bd}\gamma^{ac}$.

The step (b) uses the properties $\omega_\mu{}^{mn} = -\omega_\mu{}^{nm}$ and $\lambda_{ab} = -\lambda_{ba}$.

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